LECTURES ON ALGEBRAIC CYCLES AND CHOW GROUPS

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Contents

Based on lectures at the *Summer School on Hodge Theory and Related Topics* organized by E. Cattani, F. El Zein, P. Griffiths and Lê Dung Tràng at the ICTP of Trieste from 14th June to to 2nd July 2010.

These are the notes of my lectures in the ICTP Summer School and conference on “Hodge Theory and Related Topics” 2010.

The notes are informal and close to the lectures themselves. As much as possible I have concentrated on the main results. Specially in the proofs I have tried to outline the main ideas and mostly omitted the technical details. In order not to “wave hands” I have often written “outline or indication of proof” instead of “proof”; on the other hand when possible I have given references where the interested reader can find the details for a full proof.

The first two lectures are over an arbitrary field (for simplicity always assumed to be algebraically closed), lectures III and IV are over the complex numbers and in lecture V we return to an arbitrary field.

I have tried to stress the difference between the theory of divisors and the theory of algebraic cycles of codimension larger than one. In lectures IV and V, I have discussed results of Griffiths and Mumford which -to my opinion- are the two most striking facts which make this difference clear.

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1. Lecture I: Algebraic Cycles. Chow groups

1.1. **Assumptions and conventions.** In the first two lectures k is an *algebraically closed field*. We work with *algebraic varieties* defined over k (i.e. $k-$schemes which are reduced, i.e. there are no nilpotent elements in the structure sheaves). We *assume moreover* (unless otherwise stated) that our varieties are *smooth, quasi-projective and irreducible*. We denote the category of such varieties by $\mathrm{Var}(k)$ (the morphisms are the usual morphisms, i.e. rational maps which are everywhere regular). (See [Har, Chap. 1]). If X is such a variety, let $d = \dim X$; in the following we often denote this shortly by X_d.

1.2. **Algebraic cycles.** Let $X_d \in \mathrm{Var}(k)$; let $0 \le i \le d$ and $q = d - i$. Let $\mathcal{Z}_q(X) = \mathcal{Z}^i(X)$ be the group of *algebraic cycles of dimension q (i.e. codimension i) on X*[1], i.e. the free abelian group generated by the $k-$irreducible subvarieties W on X of dimension q, but W *not* necessarily smooth. Therefore such and algebraic cycle $Z \in \mathcal{Z}_q(X) = \mathcal{Z}^i(X)$ can be written as $Z = \sum_\alpha n_\alpha W_\alpha$,

[1] Usually we prefer to work with the codimension i, but sometimes it is more convenient to work with the dimension q.

a finite sum with $n_\alpha \in \mathbb{Z}$ and $W_\alpha \subset X$ $q-$dimensional subvarieties of X defined over k and irreducible but not necessarily smooth.

Example 1.1.

a. $\mathcal{Z}^1(X) = \mathrm{Div}(X)$ is the group of (Weil) *divisors* on X.

b. $\mathcal{Z}_0(X) = \mathcal{Z}^d(X)$ is the group of $0-$cycles on X, so $Z \in \mathcal{Z}_0(X)$ is a formal sum $Z = \sum n_\alpha P_\alpha$ with $P_\alpha \in X$ points. Put $\deg(Z) = \sum n_\alpha$.

c. $\mathcal{Z}_1(X) = \mathcal{Z}^{d-1}(X)$ is the group of *curves* on X, i.e. $Z = \sum n_\alpha C_\alpha$ with $C_\alpha \subset X$ curves.

1.2.1. *Operations on algebraic cycles.* There are three *basic* operations and a number of other operations which are built from these basic operations:

1. **Cartesian product** If $W \subset X_1$ (resp. $V \subset X_2$) is a subvariety of dimension q_1 (resp. q_2) then $W \times V \subset X_1 \times X_2$ is a subvariety of dimension $q_1 + q_2$. Proceeding by linearity we get

$$\mathcal{Z}_{q_1}(X_1) \times \mathcal{Z}_{q_2}(X_2) \longrightarrow \mathcal{Z}_{q_1+q_2}(X_1 \times X_2).$$

2. **Push-forward** (See [F, p. 11]) Given a morphism $f : X \to Y$ we get a homomorphism $f_* : \mathcal{Z}_q(X) \to \mathcal{Z}_q(Y)$. By linearity it suffices to define this only for a subvariety $W \subset X$. Now consider the *set-theoretical* image $f(W) \subset Y$, its Zariski closure $\overline{f(W)}$[2] is an algebraic subvariety of Y, irreducible if W itself is irreducible and $\dim \overline{f(W)} \leq \dim W = q$. Now define

$$f_*(W) = \begin{cases} 0 & \text{if } \dim \overline{f(W)} < \dim W \\ [k(W) : k(\overline{f(W)})] \cdot \overline{f(W)} & \text{if } \dim \overline{f(W)} = \dim W \end{cases}$$

where $k(W)$ is the function field of W (i.e. the field of rational functions on W), similarly $k(\overline{f(W)})$ is the function field of $\overline{f(W)}$ (and note that we have a finite extension of fields in the case $\dim \overline{f(W)} = \dim W$).

3. **Intersection product** (only defined under a *restriction*!) Let $V \subset X$ (resp. $W \subset X$) be an irreducible subvariety of codimension i (resp. j). Then $V \cap W$ is a finite union $\bigcup A_l$ of irreducible subvarieties $A_l \subset X$. Since X is smooth all A_l have codimension $\leq i + j$ ([Har, p. 48], [F, p. 120]).

Definition 1.2. The intersection of V and W at A_l is called *proper* (or *good*) if codimension of A_l in X is $i + j$.

In that case we define the *intersection multiplicity* $i(V \cdot W; A_l)$ of V and W at A_l. This intersection multiplicity is defined as follows:

Definition 1.3. (See [Har, p. 427] and/or [S, p. 144])

$$i(V \cdot W; A_l) := \sum_{r=0}^{\dim X} (-1)^r \mathrm{length}_{\mathcal{O}}\{\mathrm{Tor}_r^{\mathcal{O}}(\mathcal{O}/J(V), \mathcal{O}/J(W))\}.$$

Here $\mathcal{O} = \mathcal{O}_{A_l,X}$ is the local ring of A_l in X and $J(V)$ (resp. $J(W)$) is the ideal defining V (resp. W) in $\mathcal{O}$.

[2] Or $f(W)$ itself if f is proper.

Now if the intersection is proper at every A_l then one defines the *intersection product as a cycle* by

$$V \cdot W := \sum_l i(V \cdot W; A_l) A_l.$$

This is an algebraic cycle in $Z^{i+j}(X)$.

By linearity one defines now in an obvious way the intersection product $Z_1 \cdot Z_2 \in \mathcal{Z}^{i+j}(X)$ for $Z_1 = \sum n_\alpha V_\alpha \in \mathcal{Z}^i(X)$ and $Z_2 = \sum m_\beta W_\beta \in \mathcal{Z}^j(X)$ as:

$$Z_1 \cdot Z_2 = \sum n_\alpha m_\beta (V_\alpha \cdot W_\beta).$$

Remark 1.4.

a. For the notion of length of a module see -for instance- [F, p.406].
b. For the intersection multiplicity $i(V \cdot W; A_l)$ one could try -more naively- to work *only* with the tensor product of $\mathcal{O}/J(V)$ and $\mathcal{O}/J(W)$ but this is not correct (see [Har, p. 428, ex. 1.1.1]). One needs for correction the terms with the Tor's. The Tor-functos are the so-called "higher derived functors" for the tensor product functor. See for instance page 159 in the book [E, p.159] of Eisenbud or chapters III and IV in the book [HS] of Hilton and Stammbach.
c. The above definition of intersection multiplicity of Serre coincides with the older and more geometric definitions of Weil, Chevalley and Samuel (see [S, p. 144]).

Now we discuss *further operations* on algebraic cycles built via the basic operations.

4. **Pull-back of cycles** (not always defined!) Given a morphism $f : X \to Y$ we want to define a homomorphism $f^* : \mathcal{Z}^i(Y) \to \mathcal{Z}^i(X)$. So let $Z \in \mathcal{Z}^i(Y)$.

Definition 1.5. $f^*(Z) := (\mathrm{pr}_X)_*(\Gamma_f \cdot (X \times Z))$, where Γ_f is the graph of f.

But this is only defined if the intersection $\Gamma_f \cdot (X \times Z)$ is defined[3].

Remark 1.6. This is defined if $f : X \to Y$ is *flat* (see [F, p. 18]). This happens in particular if $X = Y \times Y'$ and f is the projection on Y.

5. **Correspondences and operations of correspondences on algebraic cycles.** Let $X_d, Y_e \in \mathrm{Var}(k)$. A *correspondence* $T \in \mathrm{Cor}(X, Y)$ from X to Y is an element $T \in \mathcal{Z}^n(X \times Y)$ for a certain $n \geq 0$, i.e. $\mathrm{Cor}(X, Y)$ equals $\mathcal{Z}(X \times Y)$. We denote the transpose by ${}^tT \in \mathcal{Z}^n(X \times Y)$, so ${}^tT \in \mathrm{Cor}(Y, X)$. Given $T \in \mathcal{Z}^n(X \times Y)$ then we define the homomorphism

$$T : \mathcal{Z}^{\bullet i}(X_d) \longrightarrow \mathcal{Z}^{i+n-d}(Y_e)$$

by the formula

$$T(Z) := (\mathrm{pr}_Y)_*\{T \cdot (Z \times Y)\},$$

[3]The intersecion is of course playing on $X \times Y$

but this is *only defined* on a *subgroup* $\mathcal{Z}^{\bullet i} \subset \mathcal{Z}^i(X)$, namely on those Z for which the intersection product $T \cdot (Z \times Y)$ is defined (on $X \times Y$).

Remark 1.7. If we have a morphism $f : X \to Y$ then for $T = \Gamma_f$ we get back f_* and for $T = {}^t\Gamma_f$ we get f^*.

1.3. **Adequate equivalence relations.** It will be clear from the above that one wants to introduce on the group of algebraic cycles a "good" equivalence relation in such a way that -in particular- the above operations are *always defined* on the corresponding *cycle classes*.

Samuel introduced in 1958 the notion of *adequate* (or "good") *equivalence relation* ([Sam, p. 470]). Roughly speaking an equivalence relation is adequate if it is compatible with addition and intersection and if it is functorial. The precise conditions are as follows.

An *equivalence relation* $\sim$ given on the groups of algebraic cycles $Z(X)$ of all varieties $X \in \mathrm{Var}(k)$ is *adequate* if it satisfies the following conditions:

- (R1) $\mathcal{Z}^i_\sim(X) := \{Z \in \mathcal{Z}^i(X) : Z \sim 0\} \subset \mathcal{Z}^i(X)$ is a subgroup.
- (R2) If $Z \in \mathcal{Z}^i(X)$, $Z' \in \mathcal{Z}^i(X)$, $W \in \mathcal{Z}^j(X)$ are such that $Z \cdot W$ and $Z' \cdot W$ are defined and $Z \sim Z'$ then also $Z' \cdot W \sim Z \cdot W$.
- (R3) Given $Z \in \mathcal{Z}^i(X)$ and a finite number of subvarieties $W_\alpha \subset X$ then there exists $Z' \in \mathcal{Z}^i(X)$ such that $Z' \sim Z$ and such that all $Z' \cdot W_\alpha$ are defined.
- (R4) Let $Z \in \mathcal{Z}(X)$ and $T \in Z(X \times Y)$ be such that $T \cdot (Z \times Y)$ is defined. Assume that Y is proper (for instance projective) and that $Z \sim 0$. Then also $T(Z) \sim 0$ in $\mathcal{Z}(Y)$. Recall $T(Z) = (\mathrm{pr}_Y)_*(T \cdot (Z \times Y))$.

Now let $\sim$ be an *adequate* equivalence relation for algebraic cycles. Put $C^i_\sim(X) := \mathcal{Z}^i(X)/\mathcal{Z}^i_\sim(X)$, (and similarly $C^\sim_q(X)$ if $q = d - i$ with $d = \dim X$ if we want to work with dimension instead of codimension). Then we have:

Proposition 1.8.

a. $C_\sim(X) = \bigoplus_{i=0}^d C^i_\sim(X)$ *is a commutative ring with respect to the intersection product.*

b. *If $f : X \to Y$ is proper then $f_* : C^\sim_q(X) \to C^\sim_q(Y)$ is an additive homomorphism.*

c. *If $f : X \to Y$ (arbitrary!) then $f^* : C_\sim(X) \to C_\sim(X)$ is a ring homomorphism.*

Proof. Left to the reader (or see [Sam]). Hint: a. and b. are straightforward. For c. one can use the "reduction to the diagonal". Namely if Z_1 and Z_2 are algebraic cycles on X such that $Z_1 \cdot Z_2$ is defined then $Z_1 \cdot Z_2 = \Delta_*(Z_1 \times Z_2)$ where $\Delta : X \hookrightarrow X \times X$ is the diagonal (see [S, V-25]). □

Proposition 1.9 (Supplement)**.** *Let $T \in \mathcal{Z}(X \times Y)$. Then T defines an additive homomorphism $T : C_\sim(X) \to C_\sim(Y)$ and this homomorphism depends only on the class of T in $\mathcal{Z}(X \times Y)$.*

Proof. For the definition of T as operator on the cycles see section 1.2 above. For the proof see [Sam, prop. 7, p. 472]. □

We shall discuss in the remaining part of this lecture I and in lecture II the following adequate equivalence relations:

a. Rational equivalence (Samuel and Chow independently, 1956)

b. Algebraic equivalence (Weil, 1952)

c. Smash-nilpotent equivalence (Voevodsky, 1995)

d. Homological equivalence

e. Numerical equivalence

Homological (at least if $k = \mathbb{C}$) and numerical equivalence are kind of classical and the origin is difficult to trace.

1.4. **Rational equivalence. Chow groups.** Rational equivalence, defined and studied independently in 1956 by Samuel and Chow, is a *generalization* of the classical concept of *linear equivalence for divisors.*

1.4.1. *Linear equivalence for divisors.* Let $X = X_d$ be an irreducible variety but for the moment (for technical reasons, see section 1.4.2) *not* necessarily smooth. Let $\varphi \in k(X)^*$ be a rational function on X. Recall [F, p. 8]

$$\mathrm{div}(\varphi) := \sum_{\substack{Y \subset X \\ \text{codim } 1}} \mathrm{ord}_Y(\varphi) \cdot Y.$$

Here Y "runs" through the irreducible subvarieties of codimension one and $\mathrm{ord}_Y(\varphi)$ is defined as:

a. if $\varphi \in \mathcal{O}_{Y,X}$ then $\mathrm{ord}_Y(\varphi) := \mathrm{length}_{\mathcal{O}_{Y,X}}(\mathcal{O}_{Y,X}/(\varphi))$

b. otherwise write $\varphi = \varphi_1/\varphi_2$ with $\varphi_1, \varphi_2 \in \mathcal{O}_{Y,X}$ and

$$\mathrm{ord}_Y(\varphi) := \mathrm{ord}_Y(\varphi_1) - \mathrm{ord}_Y(\varphi_2)$$

(this is well defined!)

Remark 1.10. If X is smooth at Y then $\mathcal{O}_{Y,X}$ is a discrete valuation ring and $\mathrm{ord}_Y(\varphi) = \mathrm{val}_Y(\varphi)$.

So (always) $\mathrm{div}(\varphi)$ is a Weil-divisor and put $\mathrm{Div}_l(X) \subset \mathrm{Div}(X)$ for the subgroup generated by such divisors; in fact

$$\mathrm{Div}_l(X) = \{D = \mathrm{div}(\varphi);\ \varphi \in k(X)^*\}$$

and $\mathrm{CH}^1(X) := \mathrm{Div}(X)/\mathrm{Div}_l(X)$ is the (Chow) group of the *divisor classes* with respect to linear equivalence.

1.4.2. *Rational equivalence. Definition.* Let $X = X_d \in \mathrm{Var}(k)$, i.e. smooth, quasi-projective and irreducible of dimension d. Let $0 \leq i \leq d$ and put $q = d - i$.

Definition 1.11. $\mathcal{Z}_q^{rat}(X) = \mathcal{Z}_{rat}^i(X) \subset \mathcal{Z}^i(X)$ is the subgroup generated by the algebraic cycles of type $Z = \mathrm{div}(\varphi)$ with $\varphi \in k(Y)^*$ with $Y \subset X$ an irreducible subvariety of codimension $(i-1)$ (i.e. of dimension $(q+1)$) (see [F, chap. 1], in particular page 10). Note that we do *not* require Y to be smooth.

Remark 1.12.

a. Equivalently: let $Z \in \mathcal{Z}_q(X)$. $Z \sim_{rat} 0$ if and only if there exists a finite collection $\{Y_\alpha, \varphi_\alpha\}$ with $Y_\alpha \subset X$ irreducible and of dimension $(q+1)$ and $\varphi_\alpha \in k(Y_\alpha)^*$ such that $Z = \sum_\alpha \mathrm{div}(\varphi_\alpha)$.

b. We do *not* assume that the Y_α are smooth, therefore it is important that $\mathrm{div}(\varphi)$ is defined for non zero rational functions on *arbitrary* varieties.

c. Clearly $Z_{rat}^1(X) = \mathcal{Z}_{lin}^1(X) = \mathrm{Div}_l(X)$, i.e. for divisors rational equivalence is linear equivalence.

There is another equivalent formulation [F, p.15] for rational equivalence (which was in fact used in the original definition by Samuel and by Chow). Namely:

Proposition 1.13. *Let $Z \in \mathcal{Z}^i(X)$. The following conditions are equivalent:*

a. *Z is rationally equivalent to zero.*

b. *There exists a correspondence $T \in \mathcal{Z}^i(\mathbb{P}^1 \times X)$ and two points $a, b \in \mathbb{P}^1$ such that $Z = T(b) - T(a)$*[4]*.*

Proof. The first implication is easy. Assume for simplicity $Z = \mathrm{div}(\varphi)$ with $\varphi \in k(Y)^*$ and $Y \subset X$ an irreducible subvariety of dimension $(q+1)$, then take $T =^t \Gamma_\varphi$ on $\mathbb{P}^1 \times Y$ and consider it as cycle on $\mathbb{P}^1 \times Y$ via $\iota : Y \hookrightarrow X$ (so strictly speaking $T = (\mathrm{id} \times \iota)_*({}^t\Gamma_\varphi)$). If $Z = \sum \mathrm{div}(\varphi_\alpha)$ do this for every φ_α.

The second implication is less easy and depends on the following theorem (see [F, prop. 1.4], also for the proof):

Theorem 1.14. *Let $f : V \to W$ be a proper, surjective morphism of irreducible varieties and $\varphi \in k(V)^*$. Then:*

a. $f_*(\mathrm{div}(\varphi)) = 0$ *if* $\dim V > \dim W$.

b. $f_*(\mathrm{div}(\varphi)) = \mathrm{div}(N(\varphi))$ *if* $\dim V = \dim W$ *where* $N = \mathrm{Norm}_{k(V)/k(W)}$.

Now for b. implies a. in proposition 1.13 we can assume that T is irreducible and $b = 0$ and $a = \infty$ on $\mathbb{P}^1$. We have on T the function φ induced by the "canonical function" t on $\mathbb{P}^1$ (i.e. $\varphi = \mathrm{pr}_{\mathbb{P}^1}^{-1}(t)$). Now apply the theorem with $V = T$ and $W = \mathrm{pr}_X(T) \subset X$ (the settheoretic projection). $\square$

[4]Recall that for $t \in \mathbb{P}^1$ we have $T(t) = (\mathrm{pr}_X)_*(T \cdot (t \times X))$.

1.4.3. *Properties of rational equivalence.* See [F, Chap. 1].

Proposition 1.15. *Rational equivalence is an adequate equivalence relation.*

Proof. (Indications only!) $(R1)$ is immediate from the definition. $(R2)$ is also easy if we use the alternative definition from the proposition 1.13 in section 1.4.2. Indeed, with some easy modifications we can get a $T \in \mathcal{Z}^i(\mathbb{P}^1 \times X)$ such that $T(a) = Z$, $T(b) = Z'$ and then from the assumptions we get that $T\cdot(\mathbb{P}^1 \times W) = T_1 \in Z^{i+j}(\mathbb{P}^1 \times X)$ is defined and $T_1(a) = Z\cdot W$ and $T_1(b) = Z'\cdot W$. The proof of $(R4)$ is left to the reader; see theorem 1.4. in [F, p.11]. The crucial property is $(R3)$. This is the so-called *Chow's moving lemma*; the origin of the idea of the proof is classical and goes back to Severi who used it for his so-called "dynamical theory" of intersection numbers. We outline the main idea; for details see [R].

We can assume that $X_d \subset \mathbb{P}^N$, that $Z \in \mathcal{Z}^i(X_d)$ is itself an irreducible subvariety and that we have only one (irreducible) $W \subset X_d$ of codimension j in X. The intersection $Z \cap W$ would be proper ("good") if all the components have codimension $i+j$ in X, so let us assume that there is a component of codimension $(i+j-e)$ with $e > 0$, e is called the *excess*, denoted by $e(Z, W)$.

Assume first that $X = \mathbb{P}^N$ itself. Let $\tau : \mathbb{P}^N \to \mathbb{P}^N$ be a projective transformation. Consider the transform $Z' = \tau(Z)$ of Z, then $Z' \sim Z$ rationally equivalent. To make this more explicite, remember that such a projective transformation is given by linear equations in the coordinates of the $\mathbb{P}^N$. Now take the coefficients occurring in these equations, they determine a point in an affine space $\mathbb{A}^M$, where $M = (N+1)^2$. So both the transformation τ and the identity transformation, say τ_0, can be considered as points in this $\mathbb{A}^M$; connect they by a line L (itself a space $\mathbb{A}^1$) and consider in $\mathbb{P}^N$ the transformations corresponding with the point $t \in L$, then the cycles $t(Z)$ give a family of cycles which determine a cycle $T \in \mathcal{Z}^i(L \times X)$ as in proposition 1.13. Taking on L the points $a = \tau$ and $b = \tau_0$ we get by proposition 1.13 that $T(\tau) = \tau(Z) = Z'$ is rational equivalent to $T(\tau_0) = Z$. Now taking τ "'sufficiently general" we can show that $\tau(Z) \cap W$ intersects properly[5].

Next consider the general case $X_d \subset \mathbb{P}^N$. Choose a linear space $L \subset \mathbb{P}^N$ of codimension $(d+1)$ and such that $X \cap L = \emptyset$. Now consider the cone $C_L(Z)$ on Z with "vertex" L. One can show (see [R]) that if we take L "sufficiently general" then $C_L(Z) \cdot X = 1 \cdot Z + Z_1$ with $Z_1 \in \mathcal{Z}^i(X)$ and where moreover the excess $e(Z_1, W) < e$. Next take again a "sufficiently general" projective transformation $\tau : \mathbb{P}^N \to \mathbb{P}^N$, then we have $Z \sim \tau(C_L(Z)) \cdot X - Z_1 =: Z_2$ rationally equivalent and moreover $e(Z_2, W) = e(Z_1, W) < e = e(Z, W)$. Hence proceeding by induction on the excess we are done. $\square$

[5]Think for instance on the simple case in which Z and W are surfaces in $\mathbb{P}^4$, then $e > 0$ iff Z and W have a curve (or curves) in common. We have to move Z such that $\tau(Z) \cap W$ consists only of points.

1.4.4. *Chow groups.* Let as before $X \in \mathrm{Var}(k)$. Define

$$\mathrm{CH}^i(X) := \mathcal{Z}^i(X)/\mathcal{Z}^i_{rat}(X), \quad \mathrm{CH}(X) := \bigoplus_{i=0}^{\dim X} \mathrm{CH}^i(X),$$

where $\mathrm{CH}^i(X)$ is the $i-$th *Chow group* of X and $\mathrm{CH}(X)$ the total Chow group.

Remark 1.16.

a. So $\mathrm{CH}^i(X) = C^i_{\sim}(X)$ if $\sim$ is rational equivalence.
b. If $d = \dim X$ and $q = d - i$ we put also $\mathrm{CH}_q(X) = \mathrm{CH}^i(X)$.
c. The Chow groups are in fact also defined in a completely similar way if X is an arbitrary variety, see [F, Chap. 1].

Since rational equivalence is an adequate equivalence relation we get (see proposition in section 1.3):

Theorem 1.17 (Chow, Samuel, 1956). *Let $X_d, Y_n \in \mathrm{Var}(k)$, i.e. smooth, projective, irreducible varieties. Then*

a. $\mathrm{CH}(X)$ *is a commutative ring (Chow ring) with respect to the intersection product.*
b. *For a proper morphism $f : X \to Y$ we have additive homomorphisms $f_* : \mathrm{CH}_q(X) \to \mathrm{CH}_q(Y)$.*
c. *For an arbitrary morphism $f : X \to Y$ we have additive homomorphisms $f^* : \mathrm{CH}^i(Y) \to \mathrm{CH}^i(X)$ and in fact a ring homomorphism $f^* : \mathrm{CH}(Y) \to \mathrm{CH}(X)$.*
d. *Let $T \in \mathrm{CH}^n(X_d \times Y_n)$ then $T_* : \mathrm{CH}^i(X) \to \mathrm{CH}^{i+n-d}(Y)$ is an additive homomorphism (depending only on the class of T).*

We mention two other important properties of Chow groups (for the easy proofs we refer to [F, Chap. 1]).

Theorem 1.18 (Homotopy property). *Let $\mathbb{A}^n$ be affine $n-$space. Consider the projection $p : X \times \mathbb{A}^n \to X$. Then*

$$p^* : \mathrm{CH}^i(X) \to \mathrm{CH}^i(X \times \mathbb{A}^n)$$

is an isomorphism ($0 \leq i \leq \dim X$).

Remark 1.19. In [F, p.22] it is only stated that p is surjective, however by taking a point $P \in \mathbb{A}^n$ we get a section $i_P : X \to X \times \mathbb{A}^n$ of p which gives the injectivity.

Theorem 1.20 (Localization sequence). *See* [F, p. 21]. *Let $\iota : Y \hookrightarrow X$ be a closed subvariety of X, let $U = X - Y$ and $j : U \hookrightarrow X$ the inclusion. Then the following sequence is exact:*

$$\mathrm{CH}_q(Y) \xrightarrow{\iota_*} \mathrm{CH}_q(X) \xrightarrow{j^*} \mathrm{CH}_q(U) \longrightarrow 0.$$

Remark 1.21. This holds for X and Y arbitrary (not necessarily smooth or projective). Recall from remark 1.16 that the definition of $\mathrm{CH}_q(X)$ for X arbitrary variety is entirely similar as in the case X smooth projective (see [F, p.10, section 1.3]

Remark 1.22 (on the coefficients). If we want to work with $\mathbb{Q}-$coefficients we write $\mathrm{CH}_{\mathbb{Q}}(X) := \mathrm{CH}(X) \otimes_{\mathbb{Z}} \mathbb{Q}$. Of course then we loose the torsion aspects!

2. Lecture II: Equivalence relations. Short survey on the results for divisors

As in lecture I we assume X_d, Y_n, etc. to be smooth, irreducible, projective varieties defined over an algebraically closed field k.

2.1. Algebraic equivalence. (Weil, 1952).

Definition 2.1. $Z \in \mathcal{Z}^i(X)$ is algebraically equivalent to zero if there exists a smooth curve C, a cycle $T \in \mathcal{Z}^i(C \times X)$ and two points $a, b \in C$ such that $Z = T(a) - T(b)$ (equivalently replacing T by $T - C \times T(b)$ we could say $T(a) = Z$, $T(b) = 0$).

Put $Z^i_{alg}(X) = \{Z \in \mathcal{Z}^i(X) : Z \sim 0 \text{ algebraically }\}$. Clearly we have inclusion $Z^i_{rat}(X) \subset Z^i_{alg}(X)$; but in general $Z^i_{rat}(X) \neq Z^i_{alg}(X)$. For instance take $X = E$ elliptic curve, and $Z = P_1 - P_2$ with $P_i \in E$ two distinct points.

Algebraic equivalence is an adequate equivalence relation (see [Sam, p. 474]). Put $\mathrm{CH}^i_{alg}(X) := \mathcal{Z}^i_{alg}(X)/\mathcal{Z}^i_{rat}(X) \subset \mathrm{CH}^i(X)$[6].

Remark 2.2.

a. We may replace C by any algebraic variety V and $a, b \in V$ smooth points.
b. Due to the theory of *Hilbert schemes* (or more elementary *Chow varieties*) we know that $\mathcal{Z}^i(X)/\mathcal{Z}^i_{alg}(X)$ is a *discrete* group.

2.2. Smash-nilpotent equivalence.

Around 1995 Voevodsky introduced the notion of *smash nilpotence* also denoted by $\otimes-$*nilpotence* (see [A, p. 21].

Definition 2.3. $Z \in \mathcal{Z}^i(X)$ is called *smash-nilpotent to zero* on X if there exists an integer $N > 0$ such that the product of N copies of Z is rationally equivalent to zero on X^N.

Let $\mathcal{Z}^i_{\otimes}(X) \subset \mathcal{Z}^i(X)$ be the subgroup generated by the cycles smash-nilpotent to zero. It can be proved that this is an adequate equivalence relation.

There is the following important theorem:

Theorem 2.4 (Voisin, Voevodsky independently)**.**

$$\mathcal{Z}^i_{alg}(X) \otimes \mathbb{Q} \subset \mathcal{Z}^i_{\otimes}(X) \otimes \mathbb{Q}$$

Proof. It goes beyond the scope of these lectures. See [V2, Chap. 11]. □

[6]Not to be confused with $C^i_{alg}(X) = \mathcal{Z}^i(X)/\mathcal{Z}^i_{alg}(X)$!

Remark 2.5. Recently B. Kahn and R. Sebastian have shown that in the above theorem inclusion is strict. For instance on the Jacobian variety $X = J(C)$ of a general curve of genus 3 the so-called Ceresa cycle $Z = C - C^-$ is $\otimes-$nilpotent to zero but not algebraically equivalent to zero (see lecture IV, after Th. 4.14).

2.3. **Homological equivalence.** Let $H(X)$ be a "good" (so-called "Weil") cohomology theory. Without going in details let us say that this means that the $H^i(X)$ are $F-$vector spaces with F a field of characteristic zero and that all the "classical" properties for cohomology hold, so in particular there are coproducts, Poincaré duality and Künneth formula hold and there is a cycle map (see below). If $\mathrm{char}(k) = 0$ then one can assume that $k \subset \mathbb{C}$ and one can take $H(X) = H_B(X_{an}, \mathbb{Q})$ or $H_B(X_{an}, \mathbb{C})$, i.e. the classical Betti-cohomology on the underlying analytic manifold X_{an}; instead of the Betti cohomology one can also take the classical de Rham cohomology on X_{an} or the algebraic de Rham cohomology with respect to the Zariski topology on X. In the case of a general field k one can take the étale cohomology $H(X) = H_{et}(X_{\bar{k}}, \mathbb{Q}_\ell)$ ($l \neq \mathrm{char}(k)$). Note that for general k one has to make the base change from k to $\bar{k}$ (in our case we assume that $k = \bar{k}$ already); for arbitrary k the $H_{et}(X, \mathbb{Q}_\ell)$ on X itself (i.e. without base change) is certainly an interesting cohomology but it is not a Weil-cohomology in general.

For a Weil-cohomology one has a *cycle map*

$$\gamma_X : \mathrm{CH}^i(X) \longrightarrow H^{2i}(X)$$

having "nice" properties, in particular, the intersection product is compatible with the cup product, i.e. if $\alpha, \beta \in \mathrm{CH}(X)$, then $\gamma_X(\alpha \cdot \beta) = \gamma_X(\alpha) \cup \gamma_X(\beta)$.

Definition 2.6. $Z \in \mathcal{Z}^i(X)$ is *homologically equivalent* to zero if $\gamma_X(Z) = 0$.

This turns out (using the "nice" properties of the cycle map) to be again an adequate equivalence relation. Put $\mathcal{Z}^i_{hom}(X) \subset \mathcal{Z}^i(X)$ for the subgroup of cycles homologically equivalent to zero.

Remark 2.7.

a. This $\mathcal{Z}^i_{hom}(X)$ depends -at least a priori- on the choice of the cohomology theory. If $\mathrm{char}(k) = 0$ then because of the comparison theorem of Artin $H_B(X_{an}, \mathbb{Q}) \otimes \mathbb{Q}_\ell \cong H_{et}(X_{\bar{k}}, \mathbb{Q}_\ell)$ one gets from the classical theory and the étale theory (for all l) the same homological equivalence. In the general case this would follow from the -still wide open- standard conjectures of Grothendieck and also from the conjecture of Voevodsky (see c. below).
b. We have $\mathcal{Z}^i_{alg}(X) \subset \mathcal{Z}^i_{hom}(X)$ as follows from the fact that two points a and b on a curve C are homologically equivalent and from the functoriality properties of the equivalence relations. For divisors we have by a *theorem of Matsusaka* that $\mathcal{Z}^1_{alg}(X) \otimes \mathbb{Q} = \mathcal{Z}^1_{hom}(X) \otimes \mathbb{Q}$, however for $1 < i < d$ we have in general that algebraic and homological equivalence are *different* by a famous *theorem of Griffiths* (see lecture IV).

c. *Smash-nilpotent equivalence* versus *homological equivalence*. Using the Künneth formula one gets that

$$\mathcal{Z}^i_{\otimes}(X) \subset \mathcal{Z}^i_{hom}(X).$$

Voevodsky *conjectures* that in fact we have equality (and this would imply -in particular- that homological equivalence would be independent of the choice of the cohomology theory). In fact he makes even the stronger conjecture that it coincides with numerical equivalence, see section 2.4 below.

2.4. **Numerical equivalence.** Let as before $X_d \in \mathrm{Var}(k)$. Let $Z \in \mathcal{Z}^i(X)$ and $W \in \mathcal{Z}^{d-i}(X)$ then their intersection product $Z \cdot W \in \mathcal{Z}_0(X)$, i.e. $Z \cdot W$, is a zero-cycle $Z \cdot W = \sum n_\alpha P_\alpha$ ($P_\alpha \in X$) and hence it has a *degree* $\sum n_\alpha$. (We can assume that $Z \cdot W$ is defined because replacing Z by Z' rationally equivalent to Z we have that $Z \cdot W$ and $Z' \cdot W$ have the same degree).

Definition 2.8. $Z \in \mathcal{Z}^i(X)$ is *numerically equivalent* to zero if $\deg(Z \cdot W) = 0$ for all $W \in \mathcal{Z}^{d-i}(X)$. Let $\mathcal{Z}^i_{num}(X) \subset \mathcal{Z}^i(X)$ be the subgroup of the cycles numerically equivalent to zero.

Numerical equivalence is an adequate equivalence relation ([Sam, p. 474])

Remark 2.9.

a. $\deg(Z \cdot W)$ is called the intersection number of Z and W and is sometimes denoted by $\sharp(Z \cdot W)$.

b. Because of the compatibility of intersection with the cup product of the corresponding cohomology classes we have

$$\mathcal{Z}^i_{hom}(X) \subseteq \mathcal{Z}^i_{num}(X).$$

For divisors we have $\mathrm{Div}_{hom}(X) = \mathrm{Div}_{num}(X)$ (theorem of Matsusaka). It is a *fundamental conjecture* that equality $\mathcal{Z}^i_{hom}(X) = \mathcal{Z}^i_{num}(X)$ should hold for all i. This is part of the standard conjectures of Grothendieck (and it is usually denoted as conjecture $D(X)$).

Remark 2.10.

a. For $k = \mathbb{C}$ the $D(X)$ would follow from the famous Hodge conjecture (see lecture III).

b. For arbitrary $k = \bar{k}$ Voevodsky *conjectures*

$$\mathcal{Z}^i_{\otimes}(X) \otimes \mathbb{Q} = \mathcal{Z}^i_{num}(X) \otimes \mathbb{Q}$$

i.e. nilpotent equivalence, homological equivalence and numerical equivalence should coincide (at least up to torsion). Of course this would imply conjecture $D(X)$.

2.5. **Final remarks and resumé of relations and notations.** There are also other interesting equivalence relations for algebraic cycles (see for instance [Sam] and [J]), however in these lectures we restrict to the above ones. One can show (see [Sam, p. 473]) that for any adequate relation $\sim$ inclusion $\mathcal{Z}^i_{rat}(X) \subset \mathcal{Z}^i_{\sim}(X)$ holds, i.e. rational equivalence is the most fine adequate equivalence.

Resumé of the relations:

$$\mathcal{Z}^i_{rat}(X) \subsetneqq \mathcal{Z}^i_{alg}(X) \subsetneqq \mathcal{Z}^i_{\otimes}(X) \subseteq \mathcal{Z}^i_{hom}(X) \subseteq \mathcal{Z}^i_{num}(X) \subset \mathcal{Z}^i(X).$$

Dividing out by rational equivalence we get the following subgroups in the Chow groups:

$$\mathrm{CH}^i_{alg}(X) \subsetneqq \mathrm{CH}^i_{\otimes}(X) \subseteq \mathrm{CH}^i_{hom}(X) \subseteq \mathrm{CH}^i_{num}(X) \subset \mathrm{CH}^i(X).$$

2.6. **Cartier divisors and the Picard group.**

2.6.1. $\mathrm{Div}(X) = \mathcal{Z}^i(X)$ is the group of the *Weil divisors*. There are also the *Cartier divisors* which are more suited if one works with an arbitrary variety (see [Har, p.140-145]). So let X be an arbitrary variety, but irreducible (and always defined over k). Let $K = k(X)$ be the function field of X and K^*_X the constant sheaf K^* on X. Let $\mathcal{O}^*_X \subset K^*_X$ be the sheaf of the units in $\mathcal{O}_X$ and define the quotient sheaf $\underline{\mathrm{Div}}_X := K^*_X/\mathcal{O}^*_X$ (always in the Zarisky topology). So we have an exact sequence:

$$1 \to \mathcal{O}^*_X \to K^*_X \to \underline{\mathrm{Div}}_X \to 1.$$

$\underline{\mathrm{Div}}_X$ is called the *sheaf of Cartier divisors* and the global sections $\Gamma(X, \underline{\mathrm{Div}}_X)$ are the *Cartier divisors*; we denote the corresponding group by $\mathrm{CaDiv}(X)$. So concretely a Cartier divisor D is given via a collection $\{U_\alpha, f_\alpha\}$ with $\{U_\alpha\}$ an open Zariski covering of X and $f_\alpha \in K$ rational functions such that f_α/f_β is in $\Gamma(U_\alpha \cap U_\beta, \mathcal{O}^*_X)$.

A Cartier divisor D is called *linear equivalent to zero* (or *principal*) if there exists $f \in K$ such that $D = \mathrm{div}(f)$, i.e. if D is in the image of

$$K^* = \Gamma(X, K^*_X) \to \Gamma(X, \underline{Div}_X)$$

and the quotient group is denoted by $\mathrm{CaCl}(X) := \Gamma(X, \underline{Div}_X)/\sim$.

Since X is irreducible we have $H^1(X, K^*_X) = H^1(X, K^*) = 1$ and the above exact sequence gives the following isomorphism (see [Har, p. 145]):

$$\mathrm{CaCl}(X) \cong H^1(X, \mathcal{O}^*_X).$$

2.6.2. *Picard group.* On a ringed space, in particular on an algebraic variety, the isomorphism classes of invertible sheaves form an *abelian group* under the tensor product, the so-called *Picard group* $\mathrm{Pic}(X)$. From the definition of invertible sheaves, i.e. from the fact that such an invertible sheaf is Zariski-locally isomorphic to $\mathcal{O}^*_X$, we get

$$\mathrm{Pic}(X) \cong H^1(X, \mathcal{O}^*_X)$$

(see [Har, p.224,ex.4.5]). Combining with the above we have

$$\mathrm{CaCl}(X) \cong H^1(X, \mathcal{O}_X^*) \cong \mathrm{Pic}(X).$$

2.6.3. *Weil divisors and the Picard group.* Let us now assume again that X is a *smooth* irreducible variety. Then for every point $P \in X$ the local ring $\mathcal{O}_{P,X}$ is a unique factorization domain and every Weil divisor is given in $\mathcal{O}_{P,X}$ by an equation f_p unique up to a unit in $\mathcal{O}_{P,X}$. From this it follows easily that Weil divisors and Cartier divisors coincide, i.e. $\mathrm{Div}(X) \xrightarrow{\sim} \mathrm{CaDiv}(X)$ (see [Har, p. 141, prop. 6.11]). Moreover, $D \in \mathrm{Div}(X)$ is of the form $\mathrm{div}(f)$, $f \in K^*$ iff D is principal. Therefore we get if X is smooth and projective

$$\mathrm{CH}^1(X) = \mathrm{CaCl}(X) = H^1(X, \mathcal{O}_X^*) = \mathrm{Pic}(X).$$

Remark 2.11*.* See [Har, p. 129, ex. 5.18(d)]. There is also a one-to-one correspondence between isomorphism classes of invertible sheaves on X and isomorphism classes of line bundles on X.

2.7. **Resumé of the main facts for divisors.** Let X be a smooth, irreducible, projective variety. In $\mathrm{CH}^1(X)$ we have the following subgroups:

$$\mathrm{CH}^1_{alg}(X) \subset \mathrm{CH}^1_{\tau}(X) \subset \mathrm{CH}^1_{hom}(X) \subset \mathrm{CH}^1_{num}(X) \subset \mathrm{CH}^1(X).$$

The following facts are known (see Mumford, appendix to Chapter V in [Zar]):

a. $\mathrm{CH}^1_{alg}(X)$ has the structure of an *abelian variety*, the so-called *Picard variety* $\mathrm{Pic}^0_{red}(X)$ (classically for $k = \mathbb{C}$ this goes back to Italian algebraic geometry, see [Zar, p. 104], in char $= p > 0$ to Matsusaka, Weil and Chow; $\mathrm{Pic}^0_{red}(X)$ is the reduced scheme of the component of the identity of the Picard scheme of Grothendieck).
b. $\mathrm{CH}^1_{\tau}(X)$ is by definition the set of divisors classes D such that $nD \sim 0$ algebraically equivalent to zero for some $n \neq 0$. By a theorem of Matsusaka (1956), $\mathrm{CH}^1_{\tau}(X) = \mathrm{CH}^1_{hom}(X) \subset \mathrm{CH}^1(X)$.
c. $\mathrm{NS}(X) := \mathrm{CH}^1(X)/\mathrm{CH}^1_{alg}(X)$ is a finitely generated (abelian) group, the so-called *Néron-Severi* group (classically Severi around 1908, in general Néron 1952).

Remark 2.12*.* For cycles of codim $i > 1$ almost all of the above facts *fail* as we shall see later (lectures IV and V). However $\mathrm{CH}^i(X)/\mathrm{CH}^i_{num}(X)$ is still, for all i and all $k = \bar{k}$, a finitely generated abelian group, as Kleiman proved in 1968 [K, Thm. 3.5, p. 379]; this follows from the existence of the Weil cohomology theory $H_{et}(X_{\bar{k}}, \mathbb{Q}_\ell)$.

Remark 2.13 (Comparison between the algebraic and the analytic theory)*.* If $k = \mathbb{C}$ we have for X a smooth projective variety several topologies, namely algebraically the Zariski topology and the étale topology, as well as the classical topology on the underlying analytic space X_{an} which is a complex manifold, compact and connected. Now there are the following comparison theorems:

a. for the étale topology (Artin, 1965):

$$H^i_{et}(X, \mathbb{Q}_\ell) \cong H^i(X_{an}, \mathbb{Q}) \otimes_{\mathbb{Q}} \mathbb{Q}_\ell.$$

b. for the Zariski topology the famous theorem of Serre (GAGA, 1956) saying that for coherent sheaves $\mathcal{F}$ the functor $\mathcal{F} \mapsto \mathcal{F} \otimes_{\mathcal{O}_{X_{alg}}} \mathcal{O}_{X_{an}} = \mathcal{F}_{an}$ is an equivalence of categories between algebraic and analytic coherent sheaves and moreover

$$H^i_{Zar}(X, \mathcal{F}) \cong H^i(X_{an}, \mathcal{F}_{an})$$
$$H^1_{Zar}(X, \mathcal{O}^*_X) \cong H^1(X_{an}, \mathcal{O}^*_{X_{an}})$$

(the latter via interpretation as invertible sheaves), so in particular $\mathrm{Pic}(X_{alg}) = \mathrm{Pic}(X_{an})$ (see Appendix B of [Har]).

Now using this GAGA theorem and the exponential exact sequence

$$0 \longrightarrow \mathbb{Z} \longrightarrow \mathcal{O}_{X_{an}} \xrightarrow{\exp} \mathcal{O}^*_{X_{an}} \longrightarrow 1,$$

most of the above *facts for divisors* become at *least plausible*. Namely from the exact exponential sequence we get the following exact sequence

$$H^1(X_{an}, \mathbb{Z}) \xrightarrow{\alpha} H^1(X_{an}, \mathcal{O}_{X_{an}}) \xrightarrow{\beta}$$
$$H^1(X_{an}, \mathcal{O}^*_{X_{an}}) \cong \mathrm{Pic}(X) \xrightarrow{\gamma} H^2(X_{an}, \mathbb{Z}).$$

Now γ is the cycle map (see later), hence $\mathrm{CH}^1(X)/\mathrm{CH}^1_{hom}(X) = \mathrm{Im}(\gamma)$ is finitely generated as subgroup of $H^2(X_{an}, \mathbb{Z})$, which is itself a finitely generated group. Next:

$$\mathrm{CH}^1_{hom}(X) \cong \mathrm{Im}(\beta) \cong H^1(X_{an}, \mathrm{O}_{X_{an}})/\mathrm{Im}(\alpha)$$

is a complex torus since $H^1(X_{an}, \mathcal{O}_{X_{an}})$ is a finite dimensional $\mathbb{C}-$vector space and $\mathrm{Im}(\alpha)$ is a lattice in this vector space (see lecture III).

2.8. **References for lectures I and II.** In these lectures we assume knowledge of the "'basic material" of algebraic geometry which can be found amply (for instance) in the book of Hartshorne [Har] (Chapter 1 and parts of Chapters 2 and 3). Appendix A of [Har] gives a nice introduction to algebraic cycles and Chow groups. The basic standard book for algebraic cycles and Chow groups is the book of Fulton [F], but here we have only needed mostly Chapter 1. Fulton's theory of intersection theory is much more advanced and precise; his main tool is not the moving lemma but the so-called deformation of the normal cone but this is much more technical. For the theory of Chow groups (over $\mathbb{C}$) one can also look at Chapter 9 of the book of Voisin [V2]. For the definition of the intersection multiplicities we have used Serre's approach [S, Chap. 5C].

3. Lecture III. Cycle map. Intermediate Jacobian. Deligne cohomology

In this lecture we assume that $k = \mathbb{C}$ is the field of complex numbers. Let as before X be a smooth, irreducible, projective variety now defined over $\mathbb{C}$. Then we have (see for instance [Har, Appendix B]) the underlying complex analytic space which is a complex manifold X_{an} compact and connected on which we have the classical "usual" topology. If there is no danger of confusion we shall sometimes, by abuse of notation, use the same letter for X and X_{an}.

3.1. **The cycle map.**

Proposition 3.1. *Let $X = X_d$ be a smooth, projective, irreducible variety defined over $\mathbb{C}$. Let $0 \leq p \leq d$, and put $q = d - p$. Then there exists a homomorphism, the cycle map, $\gamma_{X,\mathbb{Z}}$ (shortly $\gamma_{\mathbb{Z}}$) as follows*

$$\begin{tikzcd} \mathcal{Z}^p(X) \arrow[rr, "\gamma_{\mathbb{Z}}"] \arrow[dr] & & \mathrm{Hdg}^p(X) \\ & \mathrm{CH}^p(X) \arrow[ur, "\gamma_{\mathbb{Z}}"'] & \end{tikzcd}$$

where $\mathrm{Hdg}^p(X) \subset H^{2p}(X_{an}, \mathbb{Z})$ *is the subgroup defined as follows. Let* $j : H^{2p}(X_{an}, \mathbb{Z}) \to H^{2p}(X_{an}, \mathbb{C})$ *be the natural map and*

$$H^{2p}(X_{an}, \mathbb{C}) = \bigoplus_{r+s=2p} H^{r,s}(X_{an})$$

the Hodge decomposition then

$$\mathrm{Hdg}^p(X) := H^{2p}(X_{an}, \mathbb{Z}) \cap j^{-1}(H^{p,p}(X_{an})).$$

Remark 3.2. By abuse of language we have denoted the factorization of the map $\gamma_{\mathbb{Z}} : \mathcal{Z}^p(X) \to \mathrm{Hdg}^p(X)$ through the Chow group by the same symbol $\gamma_{\mathbb{Z}}$. Also it would be more correct to write $\mathrm{Hdg}^p(X_{an})$.

Remark 3.3. We shall only construct the map for $\mathcal{Z}^p(X)$ itself. The factorization exists because $\mathcal{Z}_{rat}(\cdot) \subseteq \mathcal{Z}_{hom}(\cdot)$ and this is true since it is trivially true for $X = \mathbb{P}^1$ and true in general via functoriality[7].

3.1.1. *Outline of the construction of the $\gamma_{\mathbb{Z}}$.* (For details see [V1, 11.1.2]). Let $Z_q \subset X_d$ be a closed subvariety. There is (in the analytic topology) the following exact sequence with $U = X - Z$:

$$\cdots \to H^{2p-1}(U; \mathbb{Z}) \to H^{2p}(X, U; \mathbb{Z}) \xrightarrow{\rho} H^{2p}(X; \mathbb{Z}) \to H^{2p}(U, \mathbb{Z}) \to \cdots$$

Assume for simplicity that Z is also smooth. By a theorem of Thom (see [V1, 11.1.2]) we have an isomorphism

$$T : H^{2p}(X, U; \mathbb{Z}) \xrightarrow{\cong} H^0(Z; \mathbb{Z}) = \mathbb{Z}.$$

[7] Recall that we have in fact the stronger inclusion $\mathcal{Z}_{alg}(\cdot) \subset \mathcal{Z}_{hom}(\cdot)$; see lecture III, section 2.3, remark 2.7.

Now take $\gamma_{\mathbb{Z}}(Z) = \rho \circ T^{-1}(1_{\mathbb{Z}})$. If Z is not smooth we replace it by $Z - Z_{sing}$ in the above sequence (for details see [V1, 11.1.2]).

Remark 3.4. If the variety X is defined over an algebraically closed field k but otherwise of arbitrary characteristic we have essentially the same construction (see [M, p. 268]) working with $H_{et}(X, \mathbb{Z}_\ell)$, where $\ell \neq \mathrm{char}(k)$, and using instead of the above sequence the sequence

$$\cdots \to H^{2p}_Z(X, \mathbb{Z}_\ell) \to H^{2p}(X, \mathbb{Z}_\ell) \to H^{2p}(U, \mathbb{Z}_\ell) \to \cdots$$

3.1.2. *Position of $\gamma_{\mathbb{Z}}(Z)$ in the Hodge decomposition.* For this we must use the *de Rham interpretation* of the cohomology. Recall (see [Gr-H, p. 44]) that there exists an isomorphism

$$H^i_{dR}(X_{an}, \mathbb{C}) \xrightarrow{\cong} H^i_{sing}(X_{an}, \mathbb{C}) \cong H^{sing}_i(X_{an}, \mathbb{C})^*$$

where $H_{sing}(\cdot)$ is the singular cohomology and $*$ is the dual, given by

$$\varphi \longmapsto \langle \sigma, \varphi \rangle := \int_\sigma \varphi$$

where φ is a closed $\mathcal{C}^\infty$-differential form of degree i and σ is a differentiable $i-$chain. Moreover in terms of the de Rham cohomology the Poincaré duality is given by the pairing

$$\begin{aligned} H^i_{dR}(X_{an}, \mathbb{C}) \times H^{2d-i}_{dR}(X_{an}, \mathbb{C}) &\longrightarrow \mathbb{C} \\ (\alpha, \beta) &\longmapsto \langle \alpha, \beta \rangle := \int_X \alpha \wedge \beta \end{aligned}$$

and by this pairing

$$H^{r,s}(X_{an}) \xrightarrow{\sim} H^{d-r,d-s}(X_{an})^*.$$

Returning to $\gamma_{\mathbb{Z}}(Z)$ we have

Lemma 3.5. $j \circ \gamma_{\mathbb{Z}}(Z) \in H^{p,p}(X_{an}) \subset H^{2p}(X_{an}, \mathbb{C})$

Indication of the proof. It is possible (see [V1, 11.1.2]) to choose for $j \circ \gamma_{\mathbb{Z}}(Z)$ a "de Rham representative" such that we have ([V1, Ibid]):

$$\int_X j \circ \gamma_{\mathbb{Z}}(Z) \wedge \beta = \int_Z \beta_{|Z} = \int_Z i^*(\beta)$$

where $i : Z \hookrightarrow X$. However, since Z is a $q-$dimensional complex manifold this is zero unless β is of type $(q, q) = (d-p, d-p)$. Hence $j \circ \gamma_{\mathbb{Z}}(Z) \in H^{p,p}(X_{an})$. □

3.2. **Hodge classes. Hodge conjecture.**

3.2.1. Recall (see proposition 3.1)

$$\mathrm{Hdg}^p(X) := \{\eta \in H^{2p}(X_{an}, \mathbb{Z}) : \ j(\eta) \in H^{p,p}(X_{an})\}$$

where $j : H^{2p}(X_{an}, \mathbb{Z}) \to H^{2p}(X_{an}, \mathbb{C})$ is the natural map. The elements of $\mathrm{Hdg}^p(X)$ are called *Hodge classes* or "*Hodge cycles*" of type (p, p). So we have seen in section 3.1:

Theorem 3.6. *The cohomology classes $\gamma_{\mathbb{Z}}(Z)$ of the algebraic cycles $\mathcal{Z}^p(X)$ are Hodge classes of type (p,p), i.e.*

$$\gamma^p_{\mathbb{Z},X} : \mathcal{Z}^p(X) \longrightarrow \mathrm{Hdg}^p(X) \subset H^{2p}(X_{an}, \mathbb{Z}).$$

Of course there comes up immediately the question: what is the image?

3.2.2. *Lefschetz* $(1,1)-$*theorem.* For *divisors* there is the famous

Theorem 3.7 (Lefschetz $(1,1)$, 1924)**.** *Let X be a smooth, irreducible, projective variety defined over $\mathbb{C}$. Then*

$$\gamma^1_{\mathbb{Z},X} : \mathrm{Div}(X) \longrightarrow \mathrm{Hdg}^1(X) \subset H^2(X_{an}, \mathbb{Z})$$

is onto, i.e. every Hodge class of type $(1,1)$ *is "algebraic" (i.e. is the cohomology class of a divisor).*

Indication of the proof. See [Gr-H, p. 163] for details. First using the GAGA theorems one goes from the algebraic theory to the analytic theory, namely $H^i(X, \mathcal{O}_X) \cong H^i(X_{an}, \mathcal{O}_{X_{an}})$ and $\mathrm{CH}^1(X) \cong H^1(X, \mathcal{O}^*_X) \cong H^1(X_{an}, \mathcal{O}^*_{X_{an}})$. Therefore we have to see that "the cycle map" $H^1(X_{an}, \mathcal{O}^*_{X_{an}}) \to H^2(X_{an}, \mathbb{Z})$ maps *onto* $\mathrm{Hdg}^1(X) \subset H^2(X_{an}, \mathbb{Z})$.

For this one uses the exponential sequence

$$0 \to \mathbb{Z} \to \mathcal{O}_{X_{an}} \xrightarrow{\exp} \mathcal{O}^*_{X_{an}} \longrightarrow 1$$

from which one gets an exact sequence

$$H^1(X_{an}, \mathcal{O}^*_{X_{an}}) \xrightarrow{\alpha} H^2(X_{an}, \mathbb{Z}) \xrightarrow{\beta} H^2(X_{an}, \mathcal{O}_{X_{an}}).$$

Now one shows that under the identification the boundary map α corresponds to the cycle map $\gamma_{\mathbb{Z}} : \mathrm{CH}^1(X) \to H^2(X_{an}, \mathbb{Z})$ and the map β is the "projection" of the image of $j : H^2(X_{an}, \mathbb{Z}) \to H^2(X_{an}, \mathbb{C})$ to $H^{0,2}(X_{an})$. From these identifications *the* theorem follows from the fact that $\mathrm{Im}(\alpha) = \ker(\beta)$ and because $H^{2,0} = H^{0,2}$ and therefore $\ker(\beta) = \mathrm{Im}(j) \cap H^{1,1}$ where $j : H^1(X_{an}, \mathbb{Z}) \to H^2(X_{an}, \mathbb{C})$. □

Remark 3.8. This is the modern proof due to Kodaira-Spencer (1953). For a discussion of the ideas of the original proof of Lefschetz see the very interesting paper of Griffiths in Amer. J. of Math. **101** (1979).

3.2.3. *Hodge conjecture.* Motivated by the Lefschetz $(1,1)$ theorem for divisors Hodge conjectured that, or at least raised the question whether, $\gamma_{\mathbb{Z}}$ is onto always for all p ("integral Hodge conjecture"). However Atiyah-Hirzebruch discovered that this integral form is *not* true (1962), later other counterexamples were given by Kollar (1992) and Totaro (1997). Therefore the question has to be modified to rational coefficients.

Conjecture 3.9 (Hodge)**.** $\gamma_{\mathbb{Q}} : \mathcal{Z}^p(X) \otimes \mathbb{Q} \longrightarrow \mathrm{Hdg}^p(X) \otimes \mathbb{Q}$ *is onto.*

This fundamental conjecture is wide open and only known for special cases (see for instance lectures by Murre and van Geemen in [GMV]).

Remark 3.10. In fact Hodge raised an even more general question (see Hodge, *Harmonic Integrals*, p. 214) known under the name "generalized Hodge conjecture". However in 1969 Grothendieck pointed out that this generalized Hodge conjecture *as stated by Hodge* is not true and he corrected the statement. For this GHC (Grothendieck-Hodge conjecture) see [V1, 11.3.2] or [PS, p. 164].

3.3. **Intermediate Jacobian and Abel-Jacobi map.**

3.3.1. *Intermediate Jacobian (of Griffiths).* Let X ve a smooth, irreducible, projective variety defined over $\mathbb{C}$. Recall the *Hodge decomposition*

$$H^i(X,\mathbb{C}) = \bigoplus_{r+s=i} H^{r,s}(X), \quad H^{s,r}(X) = \overline{H^{r,s}(X)}$$

(now write, by abuse of language, $X = X_{an}$) and the corresponding descending *Hodge filtration*

$$F^jH^i(X,\mathbb{C}) = \bigoplus_{r\geq j} H^{r,i-r} = H^{i,0} + H^{i-1,1} + \cdots + H^{j,i-j}.$$

Definition 3.11. The $p-$th *intermediate Jacobian* of X is

$$J^p(X) = H^{2p-1}(X,\mathbb{C})/F^pH^{2p-1}(X,\mathbb{C}) + H^{2p-1}(X,\mathbb{Z}).$$

So writing $V = H^{p-1,p} + \cdots + H^{0,2p-1}$ we have that $J^p(X) = V/H^{2p-1}(X,\mathbb{Z})$ (where -of course- we mean the image of $H^{2p-1}(X,\mathbb{Z})$ in V).

Lemma 3.12. *$J^p(X)$ is a complex torus of dimension half the $(2p-1)-th$ Betti number of X:*

$$\dim J^p(X) = \frac{1}{2}B_{2p-1}(X),$$

and hence for the conjugate $\overline{V}$ of V we have: $\overline{V} = H^{p,p-1} + \cdots + H^{2p-1,0}$ and also that the Betti number is even.

Proof. First note that due to $H^{s,r}(X) = \overline{H^{r,s}(X)}$ the Betti number is even, so let $B_{2p-1}(X) = 2m$. We have to show that the image of $H^{2p-1}(X,\mathbb{Z})$ is a *lattice* in the complex vector space V. Therefore if $\alpha_1,\ldots,\alpha_{2m}$ is a $\mathbb{Q}-$basis of $H^{2p-1}(X,\mathbb{Q})$ we must show that if $\omega = \sum r_i\alpha_i \in F^pH^{2p-1}(X,\mathbb{C})$ with $r_i \in \mathbb{Q}$ then $r_i = 0$ for all i. But $F^pH^{2p-1}(X,\mathbb{C}) = \overline{V}$ and $\omega = \overline{\omega}$ so $\omega \in V \cap \overline{V} = (0)$. Therefore $\omega = 0$, but $\{\alpha_i\}$ is a $\mathbb{Q}-$basis for $H^{2p-1}(X,\mathbb{Q})$, hence all $r_i = 0$. $\square$

Remark 3.13. The complex torus $J^p(X)$ is *in general not* an *abelian variety*, i.e. can not be embedded in projective space. For a torus $T = V/L$ to be an abelian variety it is necessary and sufficient that there exists a so-called *Riemann form*. This is a $\mathbb{R}-$bilinear form $E : V \times V \to \mathbb{R}$ satisfying

a. $E(iv, iw) = E(v, w)$.

b. $E(v, w) \in \mathbb{Z}$ whenever $v, w \in L$.

c. $E(v, iw)$ symmetric and *positive definite*

In our case there is a non-degenerate form on V given by $E(v,w) = v \cup w \cup h^{d+2-2p}$, where h is the hyperplane class in $H^2(X,\mathbb{Z})$. However this form is in general not positive definite because it changes sign on the different $H^{r,s}(X)$, but it is if only one $H^{r,s}$ occurs in V, for instance only $H^{p-1,p}$.

Special cases

a. $p = 1$. Then $J^1(X) = H^1(X,\mathbb{C})/H^{1,0} + H^1(X,\mathbb{Z})$. This is the *Picard variety* of X, which is an abelian variety.
b. $p = d$. Then $J^d(X) = H^{2d-1}(X,\mathbb{C})/H^{d,d-1} + H^{2d-1}(X,\mathbb{Z})$. This is the *Albanese variety* of X, which is an abelian variety.
c. If $X = C$ is a curve then $J^1(X)$ is the so-called *Jacobian variety* of C, an abelian variety which is at the same time the Picard variety and the Albanese variety of X.

3.3.2. *Abel-Jacobi map.* Recall

$$\mathcal{Z}^p_{hom}(X) = \{Z \in \mathcal{Z}^p(X) :\ \gamma_{\mathbb{Z}}(Z) = 0\},$$

i.e. the algebraic cycles which are *homologically equivalent* to zero.

Theorem 3.14. *There exists a homomorphism* $\mathrm{AJ}^p : \mathcal{Z}^p_{hom} \to J^p(X)$ *which factors through* $\mathrm{CH}^p_{hom}(X)$. *AJ is called the Abel-Jacobi map.*

Outline of the proof. Recall that by the Poincaré duality

$$H^{2p-1}(X) \overset{\text{dual}}{\longleftrightarrow} H^{2d-2p+1}(X) \quad H^{r,s} \overset{\text{dual}}{\longleftrightarrow} H^{d-r,d-s},$$

where $d = \dim X$. Hence we have that the $\mathbb{C}-$vector space V which occurs in the description of the intermediate Jacobian $J^p(X) = V/H^{2p-1}(X,\mathbb{Z})$ is the dual of $F^{d-p+1}H^{2d-2p+1}(X,\mathbb{C})$, because

$$\begin{aligned} V = H^{p-1,p} + \cdots + H^{0,2p-1} \overset{\text{dual}}{\longleftrightarrow} & H^{d-p+1,d-p} + \cdots + H^{d,d-2p+1} \\ & = F^{d-p+1}H^{2d-2p+1}(X,\mathbb{C}). \end{aligned}$$

Therefore an element $v \in V$ is a *functional* on $F^{d-p+1}H^{2d-2p+1}(X,\mathbb{C})$. (Note that for $H^{r,s}$ we have $0 \le r, s \le d$, so the above expressions may stop "earlier").

Now let $Z \in \mathcal{Z}^p_{hom}(X)$. Since $\dim Z = d - p$, there exists a topological $(2d-2p+1)-$chain Γ such that $Z = \partial\Gamma$. Now Γ is a functional on $F^{d-p+1}H^{2d-2p+1}(X,\mathbb{C})$, because $\omega \in F^{d-p+1}H^{2d-2p+1}(X,\mathbb{C})$ is represented by a closed $\mathcal{C}^\infty-$differential form φ of degree $2d-2p+1$, and we get the functional

$$\omega \mapsto \int_\Gamma \varphi.$$

Of course we must check that this does not depend on the choice of φ in the cohomology class ω. If φ' is another choice then $\varphi' = \varphi + d\psi$, but now one can show that one can take ψ such that in ψ there occur at least $(d-p+1)$ $dz's$ (Griffiths, Bombay Coll. 1969, p. 188), therefore we get by Stokes theorem $\int_\Gamma d\psi = \int_Z \psi = 0$ because Z is a complex manifold of dimension $(d-p)$.

Hence the choice of Γ determines an element of V and hence also of the intermediate jacobian $J^p(X) = V/H^{2p-1}(X,\mathbb{Z})$. Now if we have another Γ' such that $\partial\Gamma' = Z$ then $\Gamma' - \Gamma \in H_{2d-2p+1}(X,\mathbb{Z})$, i.e. $\Gamma' - \Gamma$ is an *integral* cycle and therefore they give the same element in $J^p(X)$, this is the element $\mathrm{AJ}(Z)$.

For the fact that this factors through $\mathrm{CH}^p_{hom}(X)$ see [V1, 12.1] □

Example 3.15. Let $X = C$ be a smooth projective curve defined over $\mathbb{C}$ of genus g. Now $p = 1$ and $\mathcal{Z}^1_{hom}(C) = \mathrm{Div}^{(0)}(C)$ are the divisors of degree zero, so $D = \sum P_i - \sum Q_j$ $(P_i, Q_j \in C, 1 \leq i,j \leq m)$ so we can arrange things so that $D = \sum_i (P_i - Q_i)$. Let Γ_i be a path from Q_i to P_i, so $D = \partial\Gamma$ with $\Gamma = \sum \Gamma_i$. Now

$$J(C) = H^1(C,\mathbb{C})/(F^1H^1 + H^1(C,\mathbb{Z})) = H^{01}(C)/H^1(C,\mathbb{Z})$$

and $F^1H^1(C,\mathbb{C}) = H^{1,0}(\mathbb{C}) = H^0(C,\Omega^1_C)$, i.e. the space of holomorphic differentials, so $\dim H^{1,0} = \dim H^{0,1} = g$. Now Γ defines a functional on $H^0(C,\Omega^1_C)$, namely if $\omega \in H^0(C,\Omega^1_C)$ consider $\int_\Gamma \omega \in \mathbb{C}$. If we choose other paths (or another ordering of the points Q_j) then we get a $1-$chain Γ' and $\Gamma' - \Gamma \in H_1(C,\mathbb{Z})$ and the functionals Γ and Γ' seen as elements in the $g-$dimensional vector space $H^{01}(C)$ give the same element in $J(C) = H^{01}(C)/H^1(C,\mathbb{Z})$.

3.3.3. *The image of $\mathcal{Z}^p_{alg}(X)$ under the Abel-Jacobi map.* The intermediate jacobian behaves *functorially* under correspondences. Namely if $T \in \mathcal{Z}^p(Y_e \times X_d)$ then we get a homomorphism $T : J^r(Y) \to J^{p+r-e}(X)$. So in particular if $Y = C$ is a curve and $T \in \mathcal{Z}^p(C \times X)$ then we get a homomorphism $T : J(C) \to J^p(X)$.

The tangent space at the origin of $J^p(X)$ is the vector space used in the construction (see section 3.3.1), i.e. $V = H^{p-1,p} + \cdots + H^{0,2p-1}$. From the Künneth decomposition of the cycle class of T in $H^{2p}(C \times X)$ we see that the tangent space $H^{01}(C)$ to $J(C)$ is mapped into a subspace of $H^{p-1,p}(X) \subset V$ of the tangent space to $J^p(X)$.

Let $J^p(X)_{alg} \subseteq J^p(X)$ be the *largest subtorus* of $J^p(X)$ for which the tangent space is contained in $H^{p-1,p}(X)$. This subtorus $J^p(X)_{alg}$ is in fact an abelian variety (see remark 3.13). Of course it may happen that $J^p(X)_{alg} = 0$ (see in particular next lecture IV).

From the above it will be clear that we have

Lemma 3.16. $\mathrm{AJ}(\mathcal{Z}^p_{alg}(X)) \subseteq J^p(X)_{alg}$.

Remark 3.17. To $J^p(X)_{alg}$ corresponds a subgroup $W \subset H^{2p-1}(X,\mathbb{Z})$ which comes from the lattice which we have in the tangent space to $J^p(X)_{alg}$; in fact it is the counterimage in $H^{2p-1}(X,\mathbb{Z})$ of this lattice. This $W \subset H^{2p-1}(X,\mathbb{Z})$ is a so-called *sub-Hodge structure* of the Hodge structure

$$(H^{2p-1}(X,\mathbb{Z}), H^{2p-1}(X,\mathbb{C}) = \bigoplus H^{r,s}),$$

i.e. the Hodge structure of $H^{2p-1}(X,\mathbb{C})$ induces a Hodge-structure on $W \otimes_{\mathbb{Z}} \mathbb{C}$.

3.4. **Deligne cohomology. Deligne cycle map.**

3.4.1. *Deligne cohomology.* In this section $X = X_d$ is a smooth, irreducible, quasi-projective variety defined over $\mathbb{C}$. We denote the associated analytic space X_{an} by the same letter (X_{an} is now a complex manifold, connected but not necessarily compact). Let Ω^i_X denote the *holomorphic* differential forms of degree i (so $\Omega^0_X = \mathcal{O}_{X_{an}}$) and

$$\Omega^\bullet_X := 0 \to \Omega^0_X \xrightarrow{d} \Omega^1_X \xrightarrow{d} \Omega^2_X \to \cdots$$

is the *holomorphic de Rham complex*.

Recall that by the classical *holomorphic Poincaré lemma* ([Gr-H, p. 448]), $0 \to \mathbb{C} \to \Omega^\bullet_X$ is a resolution for $\mathbb{C}$.

Let $\Omega^{\bullet<n}_X := 0 \to \Omega^0_X \to \Omega^1_X \to \cdots \to \Omega^{n-1}_X \to 0$ be the truncated complex and $\Omega^{<n}_X[-1]$ the complex shifted one place to the right. Furthermore let $A \subset \mathbb{C}$ be a subring (usually $A = \mathbb{Z}, \mathbb{Q}$ or $\mathbb{R}$). Then Deligne considered the complex (with $A(n) = (2\pi i)^n A$)

$$A(n)^\bullet_{\mathcal{D}} : 0 \to A(n) \to \Omega^0_X \to \Omega^1_X \to \cdots \to \Omega^{n-1}_X \to 0 \to \cdots$$

in degrees zero up to n, hence

$$A(n)^\bullet_{\mathcal{D}} : 0 \to A(n) \to \Omega^{<n}_X[-1].$$

Definition 3.18 (Deligne-Beilinson cohomology with coefficients in $A(n)$)**.**

$$H^i_{\mathcal{D}}(X, A(n)) := \mathbb{H}^i(X, A(n)_{\mathcal{D}}),$$

where the $\mathbb{H}^i(X, \cdot)$ are the *hypercohomology groups* ([Gr-H, p. 445]).

More generally, if $Y \hookrightarrow X$ is a closed immersion of analytic manifolds:

Definition 3.19 (Deligne-Beilinson cohomology with support in Y)**.**

$$H^i_{Y,\mathcal{D}}(X, A(n)) := \mathbb{H}^i_Y(X, A(n)_{\mathcal{D}}).$$

Example 3.20. For $n = 0$, $H^i_{\mathcal{D}}(X, \mathbb{Z}(0)) = H^i(X, \mathbb{Z})$. Let $n = 1$. The complex $\mathbb{Z}(1)_{\mathcal{D}}$ is *quasi-isomorphic* to the complex $\mathcal{O}^*_X[-1]$ via the map $z \mapsto \exp(z)$, where $(\mathcal{O}^*_X)^\bullet$ is the complex $(\mathcal{O}^*_X)^\bullet := 1 \to \mathcal{O}^*_X \to 1$. Indeed this follows from the commutative diagram below and the exactness of the exponential sequence:

$$\begin{tikzcd}
0 \arrow[r] \arrow[d] & (2\pi i)\mathbb{Z} \arrow[r] \arrow[d, "\exp"'] & \mathcal{O}_X \arrow[r] \arrow[d, "\exp"] & 0 \arrow[d] \\
1 \arrow[r] & 1 \arrow[r] & \mathcal{O}^*_X \arrow[r] & 1
\end{tikzcd}$$

Hence we get $H^2_{\mathcal{D}}(X, \mathbb{Z}(1)) \cong H^1(X_{an}, \mathcal{O}^*_{X_{an}}) = \mathrm{Pic}(X_{an}) = \mathrm{Pic}(X_{alg})$. Now recall the exact sequence

$$0 \to \mathrm{Pic}^0(X) \to \mathrm{Pic}(X) \to \mathrm{NS}(X) \to 0 \tag{3.1}$$

and $\mathrm{Pic}^0(X) = J^1(X)$.

Example 3.21. $n = p$, the general case! $(1 \leq p \leq d = \dim X)$. The following theorem shows that the Deligne cohomology gives the following beautiful generalization of the sequence (3.1):

Theorem 3.22 (Deligne)**.** *There is an exact sequence*

$$0 \to J^p(X) \to H^{2p}_{\mathcal{D}}(X, \mathbb{Z}(p)) \to \mathrm{Hdg}^p(X) \to 0 \tag{3.2}$$

Indication of the proof. (see [V1, p.304]). The *exact sequence of complexes*

$$0 \to \Omega_X^{\bullet < p}[-1] \to \mathbb{Z}(p)_{\mathcal{D}} \to \mathbb{Z}(p) \to 0$$

gives a long exact sequence of (hyper)cohomology groups:

$$\cdots \to H^{2p-1}(X, \mathbb{Z}(p)) \xrightarrow{\alpha} \mathbb{H}^{2p}(X, \Omega_X^{\bullet < p}[-1]) \xrightarrow{\beta} H^{2p}_{\mathcal{D}}(X, \mathbb{Z}(p)) \xrightarrow{\lambda}$$
$$\xrightarrow{\lambda} H^{2p}(X, \mathbb{Z}(p)) \xrightarrow{\mu} \mathbb{H}^{2p+1}(X, \Omega_X^{<p}[-1]) \to \cdots$$

So we want to see, firstly, that $\mathrm{Im}(\beta) \cong J^p(X)$, but this amounts essentially to seeing that

$$\mathbb{H}^{2p-1}(X, \Omega_X^{\bullet < p}) \cong H^{2p-1}(X, \mathbb{C}) / F^p H(X, \mathbb{C}).$$

However this follows from the short exact sequence of complexes

$$0 \to \Omega_X^{\bullet \geq p} \to \Omega_X^{\bullet} \to \Omega_X^{\bullet < p} \to 0$$

together with the corresponding long exact sequence of hypercohomology groups plus the fact that $\mathbb{H}^{2p-1}(X, \Omega_X^{\bullet \geq p}) = F^p H^{2p-1}(X, \mathbb{C})$ (see [V1, 12.3]).

Secondly we need to see that $\ker(\mu) = \mathrm{Hdg}^p(X)$, but using the same facts as above we get that

$$\mu : H^{2p}(X, \mathbb{Z}(p)) \to H^{2p}(X, \mathbb{C}) / F^p H^{2p}(X, \mathbb{C})$$

and from this we get $\ker(\mu) \cong \mathrm{Hdg}^p(X)$. □

3.4.2. *Deligne cycle map.* Assumptions as before but assume now moreover again that X is *projective*, hence X_{an} is *compact*.

Theorem 3.23 (Deligne)**.** *There is a cycle map*

$$\gamma_{\mathcal{D}} : \mathcal{Z}^p(X) \longrightarrow H^{2p}_{\mathcal{D}}(X, \mathbb{Z}(p))$$

such that the following diagram is commutative

$$\begin{tikzcd} 0 \arrow[r] & \mathcal{Z}^p_{hom}(X) \arrow[r] \arrow[d, "\mathrm{AJ}"'] & \mathcal{Z}^p(X) \arrow[r] \arrow[d, "\gamma_{\mathcal{D}}"'] & \mathcal{Z}^p(X)/\mathcal{Z}^p_{hom}(X) \arrow[r] \arrow[d, "\gamma_{\mathbb{Z}}"'] & 0 \\ 0 \arrow[r] & J^p(X) \arrow[r] & H^{2p}_{\mathcal{D}}(X, \mathbb{Z}(p)) \arrow[r] & \mathrm{Hdg}^p(X) \arrow[r] & 0 \end{tikzcd}$$

Moreover these vertical maps factor through $\mathrm{CH}^p(X)$*.*

About the construction and proof. The construction and proof is very involved and goes beyond the scope of this lecture. We refer to [V1, 12.3.3], or to lectures of Green and Murre in [GMV]. One can also consult [E-V] or lecture 15 by El Zein and Zucker in *Topics in trascendental Algebraic Geometry* (ed. Griffiths), Ann. of Math. Studies **106**. □

3.5. **References for lecture III.** For the basics of complex algebraic geometry see the book of Griffiths-Harris [Gr-H]. The topics discussed in this lecture III are all thouroughly treated in the books [V1] and [V2] of Claire Voisin; these books are the English translation of the original French book [V]. For some of the topics one can also consult the relevant lectures by Green, Voisin and the author in [GMV] which are CIME lectures held in Torino in 1993. The Deligne-Beilinson cohomology is treated in greater detail in [E-V].

4. Lecture IV: Algebraic versus homological equivalence. Griffiths group

Recall that we have $\mathcal{Z}^i_{alg}(X) \subseteq \mathcal{Z}^i_{hom}(X)$. For divisors ($i = 1$) Matsusaka proved that $\mathrm{Div}_{alg}(X)\otimes\mathbb{Q} = \mathrm{Div}_{hom}(X)\otimes\mathbb{Q}$ (see remark 2.7 in lecture II). Also for zero-cycles ($i = d = \dim X$) we have $Z \in \mathcal{Z}_0^{alg}(X)$ if and only if $\deg Z = 0$ if and only if $Z \in \mathcal{Z}_0^{hom}(X)$.

However in 1969 Griffiths proved that there exist varieties X and $i > 1$ such that $\mathcal{Z}^i_{alg}(X)\otimes\mathbb{Q} \neq \mathcal{Z}^i_{hom}(X)\otimes\mathbb{Q}$, i.e. *for* $i > 1$ *essential new features* happen for $\mathrm{CH}^i(X)$: the theory of algebraic cycles of codimension greater than 1 is *very different* from the theory of divisors!

Theorem 4.1 (Griffiths, 1969)**.** *There exist smooth, irreducible, projective varieties of dimension* 3 *(defined over* $\mathbb{C}$*) such that* $\mathcal{Z}^2_{alg}(X)\otimes\mathbb{Q} \neq \mathcal{Z}^2_{hom}(X)\otimes\mathbb{Q}$.

Therefore it is interesting to introduce the following group, nowadays called Griffiths group:

Definition 4.2. $\mathrm{Gr}^i(X) := \mathcal{Z}^i_{hom}(X)/\mathcal{Z}^i_{alg}(X)$.

So the above theorem implies:

Corollary 4.3. *There exist smooth, irreducible, projective varieties* X *(defined over* $\mathbb{C}$*) and codimensions* $i > 1$ *such that the Griffiths group* $\mathrm{Gr}^i(X)$ *is not zero and in fact* $\mathrm{Gr}^i(X)\otimes\mathbb{Q} \neq 0$.

Griffiths uses heavily results and methods of Lefschetz, so in order to discuss this theorem we have to make some preparations.

4.1. **Lefschetz theory.** For simplicity and for the application made by Griffiths, we assume that *the base field is* $\mathbb{C}$ (although most of the results are also true for étale cohomology with $\mathbb{Q}_\ell$−coefficients, $\ell \neq \mathrm{char}(k)$, if $k = \bar{k}$).

Theorem 4.4 (Lefschetz hyperplane section theorem)**.** *Let* $V_{d+1} \subset \mathbb{P}^N$ *be a smooth, irreducible variety and* $W = V \cap H$ *a smooth hyperplane section. Then*

$$H^j(V,\mathbb{Z}) \longrightarrow H^j(W,\mathbb{Z})$$

is an isomorphism for $j < d = \dim W$ *and injective for* $j = d$.

Proof. See [Gr-H, p. 156] or [V2, 1.2.2]. □

Remark 4.5.

a. This holds also if W is a *hypersurface* section of V (use the Veronese embedding),

b. Special case: take $V = \mathbb{P}^{d+1}$ itself and $W \subset \mathbb{P}^{d+1}$ hypersurface. Then we get $H^j(W, \mathbb{Z}) = 0$ if $j < \dim W$ and $H^{2j}(W, \mathbb{Z}) = \mathbb{Z}\cdot h^j$ if $2j < \dim W$ where $h = \gamma_{\mathbb{Z}}(W \cap H)$, i.e. the class of the hyperplane section on W. Using Poincaré duality we get also $H^j(W, \mathbb{Z}) = 0$ if $j > \dim W$ and $H^{2j}(W, \mathbb{Z}) = \mathbb{Z}$ if $2j > \dim W$. So the only "interesting cohomology" is in $H^d(W, \mathbb{Z})$, i.e. the "middle cohomology".

c. The same results are true, using theorem 4.4, for the cohomology of *smooth, complete intersections* $W \subset \mathbb{P}^N$.

4.1.1. *Hard (strong) Lefschetz theorem.* Let $V_{d+1} \subset \mathbb{P}^N$ be smooth, irreducible. Let $W = V \cap H$ be a smooth hyperplane section. Let $h = \gamma_{\mathbb{Z},V}(W) \in H^2(V, \mathbb{Z})$. Then there is the so-called *Lefschetz operator*

$$\begin{aligned} L_V : H^j(V, \mathbb{Z}) &\longrightarrow H^{j+2}(V, \mathbb{Z}) \\ \alpha &\longmapsto h \cup \alpha \end{aligned}$$

By repeating we get (writing $n = d + 1 = \dim V$)

$$L^r : H^{n-r}(V, \mathbb{Z}) \longrightarrow H^{n+r}(V, \mathbb{Z}), \quad (0 \le r \le n).$$

Theorem 4.6 (Hard Lefschetz)**.** $L^r : H^{n-r}(V, \mathbb{Q}) \to H^{n+r}(V, \mathbb{Q})$ *is an isomorphism for all $r \le n$ (note: $\mathbb{Q}-$coefficients!)*

For V defined over $\mathbb{C}$ this is proved by Hodge theory (see [V1, 6.2.3] and [C, 5.2]). In arbitrary characteristic for $k = \bar{k}$ this holds also for $H^{\bullet}_{et}(X, \mathbb{Q}_\ell)$ and it was proved in 1973 by Deligne at the same time as the Weil conjectures.

We mention also the following

Definition 4.7 (Primitive cohomology)**.**

$$H^{n-r}_{prim}(V, \mathbb{Q}) := \ker(L^{r+1} : H^{n-r}(V, \mathbb{Q}) \to H^{n-r+2}(V, \mathbb{Q})).$$

Using the Lefschetz operator there is the so-called Lefschetz decomposition of the cohomology into primitive cohomology. Since we are not going to use this we refer only to [V1, 6.2.3].

4.1.2. *Pencils and Lefschetz pencil.* Let $V_n \subset \mathbb{P}^N$ be smooth, irreducible. Take two hyperplanes H_0 and H_1 in $\mathbb{P}^N$ and consider *the pencil* $H_t := H_0 + tH_1 (t \in \mathbb{C})$ or better $H_\lambda = \lambda_0 H_0 + \lambda_1 H_1 (\lambda = [\lambda_0 : \lambda_1] \in \mathbb{P}^1)$, then by intersecting with V we get a *pencil of hyperplane sections* $\{W_\lambda = V \cap H_\lambda\}$ on V.

Now take H_0 and H_1 "sufficiently general". Then this pencil has the following properties (see [Gr-H, p.509] or [V2, Chap. 2]):

a. There is a finite set S of points $t \in \mathbb{P}^1$ such that W_t is *smooth* outside S. Put $U := \mathbb{P}^1 - S$.

b. for $s \in S$ the W_s has only *one singular point* x and this is an "ordinary double point" (that means that in a sufficiently small analytic neighborhood of x the W_s is given analytically by a set of equations starting with transversal linear forms plus one non-degenerate quadratic form, see [V2, 2.1.1]).

Such a family of hyperplane sections is called a *Lefschetz pencil*. The *axis* of the pencil is $A := V \cap H_0 \cap H_1$. Consider $\tilde{V} = \{(x,t) \in X \times \mathbb{P}^1; x \in W_t\}$. Then $\tilde{V} \to V$ is isomorphic to the fibre $f^{-1}(t)$ and we have the following diagram, where $t \in U$:

$$\begin{tikzcd}
W := W_t \arrow[r, hook, "\iota_t"] \arrow[d] & \tilde{V}_{|U} \arrow[r, hook] \arrow[d] & \tilde{V} \cong B_A(V) \arrow[r] \arrow[d] & V \arrow[d, dotted] \\
t & U \arrow[r, hook] & \mathbb{P}^1 \arrow[r, equal] & \mathbb{P}^1
\end{tikzcd}$$

For $t \in U$ we have the Lefschetz theorem

$$\iota_t^* : H^j(V, \mathbb{Z}) \longrightarrow H^j(W_t, \mathbb{Z})$$

which is an *isomorphism* for $j < \dim W_t$ and injective for $j = \dim W_t$.

4.1.3. *Monodromy of Lefschetz pencils.* (See [V2, Chap. 3]) Recall from above that $U = \{t \in \mathbb{P}^1 : W_t \subset V\text{smooth }\}$, $S = \mathbb{P}^1 - U$ and $S = \{s_1, \ldots, s_l\}$ finite set of points. Fix $t_0 \in U$ and write $W = W_{t_0}$. Consider $\pi_1(U) = \pi_1(U, t_0)$ the fundamental group of U with base point t_0, The $\pi_1(U)$ is generated by loops $\sigma_1, \ldots, \sigma_l$, where σ_i is a loop with origin t_0 and winding one time around s_i and the σ_i are not crossing with each other (there is one relation $\sigma_l \sigma_{l-1} \cdots \sigma_2 \sigma_1 = 1$). The $\pi_1(U)$ operates on the $H^j(W, \mathbb{Q})$, but due to the Lefschetz theorems it acts trivially if $j \neq d = \dim W$ (because the $H^j(W)$ comes for $j \neq d$ from (and via) the cohomology of V). Consider the action

$$\rho : \pi_1(U) \twoheadrightarrow \Gamma := \operatorname{Im}(\rho) \subset \operatorname{Aut}(H^d(W_d, \mathbb{Q})).$$

Γ is called the *monodromy group*.

Let $\iota : W \hookrightarrow V$ and $\iota_* : H^j(W, \mathbb{Q}) \to H^{j+2}(V, \mathbb{Q})$ the induced morphism in cohomology. One defines $H^j(W, \mathbb{Q})_{van} := \ker \iota_*$. Again due to the Lefschetz theorems $H^j(W, \mathbb{Q})_{van} = 0$ for $j \neq d$.

Definition 4.8. $H^d(W_d, \mathbb{Q})_{van} = \ker(\iota_* : H^d(W, \mathbb{Q}) \to H^{d+2}(V, \mathbb{Q})$ is called the *vanishing cohomology*[8].

We have for the Lefschetz operator on W that $L_W = \iota^* \circ \iota_*$ and therefore $H^d(W, \mathbb{Q})_{van} \subseteq H^d(W, \mathbb{Q})_{prim}$; moreover there is the orthogonal decomposition $H^d(W, \mathbb{Q}) = H^d(W, \mathbb{Q})_{van} \oplus \iota^* H^d(V, \mathbb{Q})$ (see [V2, 2.3.3]).

Now the following fact is fundamental (see [V2, 3.2.3]):

[8] Because it is the subvector space of $H^d(W, \mathbb{Q})$ generated by the so-called *vanishing cycles* ([V2, 2.3.3]) but we do not need to discuss this.

Theorem 4.9 (Lefschetz)**.** *Let $\{W_t\} \subset V_{d+1}$ be a Lefschetz pencil. Then the vanishing cohomology $E = H^d(W,\mathbb{Q})_{van}$ is an irreducible $\Gamma-$module (i.e. E is irreducible under the monodromy action. So in particular the action of Γ on $H^d(W,\mathbb{Q})$ is completely reducible, i.e. $H^d(W,\mathbb{Q})$ is a direct sum of irreducible $\Gamma-$modules and E is one of them).*

4.2. **Return to Griffiths theorem.** The key point in Griffiths theorem that $\mathcal{Z}^i_{alg}$ can be different from $\mathcal{Z}^i_{hom}$ for $i > 1$ is the following:

Theorem 4.10 (Griffiths, 1969)**.** *Let $Y \subset \mathbb{P}^N$ be smooth, irreducible and defined over $\mathbb{C}$. Let $\dim Y = 2m$, and assume $H^{2m-1}(Y,\mathbb{C}) = 0$. Let $\{X_t\}_{t\in\mathbb{P}^1}$ be a Lefschetz pencil on Y. Let $t \in \mathbb{P}^1$ be very general, i.e. $t \in \mathbb{P}^1 - B$ where B is a countable set of points on $\mathbb{P}^1$ containing in particular the points $s \in \mathbb{P}^1$ for which X_s is singular. Assume $H^{2m-1}(X_t) \neq H^{m,m-1}(X_t) \oplus H^{m,m-1}(X_t)$. Finally let $Z \in \mathcal{Z}^m(Y)$ and assume that for such t as above $Z_t = Z\cdot X_t \in \mathbb{Z}^m_{alg}(X_t)$. Then Z is homologically equivalent to zero on Y, i.e. $Z \in \mathcal{Z}^m_{hom}(Y)$.*

Indication of the main points in the proof. Write shortly $X = X_t$.

Step 1. Im $\mathrm{AJ}(\mathcal{Z}^m_{alg}(X)) = 0$ in $J^m(X)$.

Proof. Consider the monodromy action by the monodromy group Γ on the cohomology $H^{2m-1}(X,\mathbb{Q})$ (note $\dim X = 2m-1$). Because of our assumptions:

$$H^{2m-1}(X,\mathbb{Q}) = H^{2m-1}(X,\mathbb{Q})_{van} + \iota^* H^{2m-1}(Y,\mathbb{Q}) = E \oplus 0 = E$$

with $E = H^{2m-1}(X,\mathbb{Q})_{van}$. Therefore by the fundamental theorem 4.9 above in section 4.1.3 the $H^{2m-1}(X,\mathbb{Q}) = E$ is an irreducible $\Gamma-$module. On the other hand $\mathrm{AJ}(\mathcal{Z}^m_{alg}(X))$ determines, via its tangent space, also an irreducible $\Gamma-$module say $H' \subset H^{2m-1}(X,\mathbb{Q}) = E$. However $H' \subset H^{m,m-1} + H^{m-1,m}$ as we have seen above (because $\mathrm{AJ}(\mathcal{Z}^m_{alg}(X)) \subset J^m_{alg}(X)$), and since by assumption $H^{m,m-1} + H^{m-1,m} \neq H^{2m-1}(X,\mathbb{Q})$ we have $H' = 0$, hence the image of $\mathrm{AJ}(\mathcal{Z}^m_{alg}(X))$ equals zero. □

Step 2. Put $U_1 = \mathbb{P}^1 - S$, where S is the finite set of points s where X_s is singular, and consider the family $J^m(X_t), t \in U_1$. These intermediate jacobians fit together to give a fiber space

$$J^m(X/U_1) := \bigcup_{t\in U_1} J^m(X_t)$$

of complex analytic tori. For each $t \in U_1$ we have an element $\mathrm{AJ}(Z_t) \in J^m(X_t)$ where $Z_t = Z\cdot X_t \in \mathcal{Z}^m_{alg}(X_t)$. These elements fit together to give a holomorphic function

$$\nu_Z : U_1 \longrightarrow J^m(X/U_1)$$

(see [V2, Thm. 7.9]. This function is a so-called *normal function.* However in our case $\nu_Z(t) = 0$ for $t \in U \subset U_1$. U is dense in U_1, hence $\nu_Z = 0$.

Step 3. $\nu_Z(t) = 0$ implies $Z \in \mathcal{Z}^m_{hom}(Y)$. The proof depends on an infinitesimal study of normal functions which goes beyond the scope of our lectures. We

refer therefore (for instance) to section 7.2 of Nagel's lecture [N] or to Voisin's lecture 7 in [GMV]. □

4.2.1. *Application (Griffiths).* Proof of Griffiths theorem that there exist varieties X such that $\mathcal{Z}^i_{alg}(X) \neq \mathcal{Z}^i_{hom}(X)$ for certain $i > 1$.

First of all we are going to construct a Lefschetz pencil that satisfies the conditions of the above theorem 4.10 in section 4.2.

Let $V_4 = V(2) \subset \mathbb{P}^5$ be a smooth *quadric* hypersurface in $\mathbb{P}^5$. On such a 4−dimensional quadric we have two families of planes $\{P\}$ and $\{P'\}$ and $H^4(V(2), \mathbb{Q}) = \mathbb{Q} \oplus \mathbb{Q}$ with as generators the linear section $V \cdot H_1 \cdot H_2 = P + P'$ ($H_i \subset \mathbb{P}^5$ hypersurfaces) and $Z = P - P'$. Now note that the intersection number $\sharp(Z \cdot Z) = -2$ hence $Z \in \mathcal{Z}^2(V)$ is *not* homologically equivalent to zero (see for these facts for instance the exercices of [V2, Chap. 2]). Take this $V(2)$ as Y in the theorem. Note $H^3(V, \mathbb{Q}) = 0$ by the Lefschetz theorem on hypersurface sections.

Next take Lefschetz pencil $\{X_t = F_t \cdot V\}$ where F_t are hypersurfaces of a certain degree r to be specified later (use the Veronese embedding of $\mathbb{P}^5$ to make them hyperplanes). Now take $X = X_t$ *very general* (in the sense explained in the theorem 4.10 in section 4.2) and consider in $H^3(X, \mathbb{C})$ the subspace $H^{3,0}(X) = H^0(X, \Omega^3_X)$. Now Ω^3_X corresponds to the linear system $\mathcal{O}_X(K_X)$ with K_X the canonical class and $K_X = \mathcal{O}_X(-4+r)$ because by the adjunction formula $K_V = (K_{\mathbb{P}^5} + V) \cdot V = \mathcal{O}_V(-4)$ and $K_X = (K_V + X) \cdot X = \mathcal{O}_X(-4+r)$. Therefore, for $r \geq 5$, the $H^{3,0}(X) \neq 0$ and $H^3(X, \mathbb{C}) \neq H^{2,1} + H^{1,2}$ as required in the theorem.

Now consider $Z_t = Z \cdot X_t = (P - P') \cdot X_t = C_t - C'_t$ in $\mathcal{Z}^2(X_t)$, where C_t and C'_t are curves on X_t. We have $Z_t = C_t - C'_t$ homologically equivalent to zero on X_t because $\sharp H \cdot (C_t - C'_t) = 0$ for a hyperplane H in $\mathbb{P}^5$ and so the cohomology class $\gamma_{\mathbb{Z}, X_t}(Z_t) = 0$ in $H^4(X_t, \mathbb{Q}) = H^2(X^*_t, \mathbb{Q})^*$ since $H^2(X_t, \mathbb{Q}) = \mathbb{Q}(H \cdot X_t) \cong \mathbb{Q}$ so $Z_t \in \mathcal{Z}^2_{hom}(X_t)$. However it follows from the theorem that $Z_t \notin \mathcal{Z}^2_{alg}(X_t)$ because that would imply $Z \in \mathcal{Z}^2_{hom}(Y) = \mathcal{Z}^2_{hom}(V(2))$ which is not the case as we have seen.

In fact the above argument works for nZ for all $n > 0$. This proves Griffiths theorem and we have for such $X = X_t$ an element in $\mathrm{Gr}^2(X_t)$ which is non zero in $\mathrm{Gr}(X) \otimes \mathbb{Q}$.

Remark 4.11. In fact this gives a non-torsion element $\mathrm{AJ}(Z_t)$ in $J^2(X)$. We have seen that the proof uses heavily the fact that $J^2(X) \neq 0$ and the theory of normal functions.

4.2.2. *Further facts.* $\mathrm{Gr}^i(X)$ is, for all i and all X, a *countable group.* This follows from the existence of the so-called *Hilbert schemes* or, less technically, from the existence of the so-called *Chow varieties.*

"Recall" that there is the following fact (Chow and van der Waerden, Math. Ann. **113** (1937), p. 692-704): Given $X \subset \mathbb{P}^N$ the algebraic cycles on X of a

fixed dimension q and a fixed degree r are *parametrized* by an *algebraic variety* $\mathrm{Ch}(X, q, r)$ (not necessarily connected!).

Since two such cycles Z and Z' which are in the same connected component of $\mathrm{Ch}(X, q, r)$ are algebraically equivalent, the $\mathrm{Ch}(X, q, r)$ itself gives only a finite number of generators to $\mathrm{Gr}^i(X)$[9].

Further examples (mentioned only, without proofs).

a.

Theorem 4.12 (Griffiths, 1969)**.** *Consider a very general* $X_3 = V(5) \subset \mathbb{P}^4$*, i.e. a very general quintic hypersurface of dimension* 3*. Such a quintic threefold contains a finite number of lines* $\{l\}$*. Let* $Z = l_1 - l_2$ *then* $\mathrm{AJ}(Z)$ *is not a torsion pont in* $J^2(X)$*,* Z *is homologically equivalent to zero but not algebraically equivalent to zero. This type of* Z *gives therefore non-zero elements in* $\mathrm{Gr}^2(X) \otimes \mathbb{Q}$ *(see* [V1, 12.2.2]*).*

b. The above result has been improved in 1983 by Clemens.

Theorem 4.13 (Clemens, 1983)**.** *For such a very general quintic hypersurface in* $\mathbb{P}^4$ *we have* $\dim_{\mathbb{Q}} \mathrm{Gr}^2(X) \otimes \mathbb{Q} = \infty$*.*

This result is extended in 2000 by Voisin to very general Calabi-Yau threefolds.

c. *Ceresa cycle*

Theorem 4.14 (Ceresa, 1983)**.** *For a very general curve* C *of genus* $g \geq 3$ *the cycle* $Z = C - C^-$ *in* $J(C)$ *is not algebraically equivalent to zero (but it is homologically equivalent to zero). Here* C^- *is the image of* C *under the map* $x \mapsto -x$ *on* $J(C)$*.*

This gives also an example of a cycle Z such that $Z \in \mathcal{Z}_1^{\otimes}(J(C))$ but $Z \notin \mathcal{Z}_1^{alg}(J(C))$ because B. Kahn and B. Sebastian proved in 2009 that on abelian varieties A of dimension 3 the $\mathcal{Z}^2_{\otimes}(A) = \mathcal{Z}^2_{hom}(A)$ (Voevodsky's conjecture!).

d. All the previous examples were on "very general varieties", i.e. varieties defined over "large" fields. However there are also examples over number fields. To my knowledge the first such example is due to B. Harris. He proved in 1983 that the Ceresa cycle $C - C^-$ is not algebraically equivalent to zero on the Jacobian $J(C)$ where C is the *Fermat curve* $X^4 + Y^4 = 1$. Hence an example over $\mathbb{Q}$!

e. All the above examples are in general "in the spirit of Griffiths", i.e. they use the (intermediate) jacobian. However Nori has proved in 1993 that there exist varieties X such that $\mathrm{Gr}^i(X) \otimes \mathbb{Q} \neq 0$ for $i > 2$ but for which $J^i(X) = 0$. See [V2, Chap. 8] or [N].

[9]But, of course, we must take into account all $r's$.

4.3. **References for lecture IV.** For the results of Griffiths and related issues see Voisin's books [V1, 12.2.2] and [V2, Chap. 3] or also her lecture 7 in [GMV]. See also J. Nagel's lecture [N]. For the original paper of Griffiths see [Gr].

5. Lecture V: The Albanese kernel. Results of Mumford, Bloch and Bloch-Srinivas

In 1969 Mumford proved a theorem which shows another important difference between the theory of divisors and the theory of algebraic cycles of larger codimension; this time it concerns zero-cycles.

In this lecture k is (again) an algebraically closed field of arbitrary characteristic (unless stated explictly otherwise) and the varieties are smooth, irreducible, projective and defined over k.

5.1. **The result of Mumford.** Let $X = X_d$ be such a variety. Recall from lecture II that

$$\mathrm{CH}^1_{alg}(X) \xrightarrow{\sim} (\mathrm{Pic}^0(X))_{red}$$

with $(\mathrm{Pic}^0(X))_{red}$ an abelian variety, the *Picard variety* of X (if $k = \mathbb{C}$, the $J^1(X)$).

For zero-cycles $\mathrm{CH}^d_{alg}(X) = \mathrm{CH}^{alg}_0(X)$ is the group of rational equivalence classes of $0-$cycles of degree 0 and we have a homomorphism (coming from the albanese map)

$$\alpha_X : \mathrm{CH}^d_{alg}(X) \twoheadrightarrow \mathrm{Alb}(X)$$

where $\mathrm{Alb}(X)$ is an abelian variety, the *Albanese variety* of X (if $k = \mathbb{C}$, $\mathrm{Alb}(X) = J^d(X)$ and α_X is the Abel-Jacobi map) and α_X is surjective but in general *not injective* (on the contrary, see further on)! Put

$$T(X) := \mathrm{CH}^d_{alb}(X) := \ker(\alpha_X)$$

the *Albanese kernel*.

Theorem 5.1 (Mumford, 1969)**.** *Let S be an algebraic surface (smooth, projective, irreducible) defined over $\mathbb{C}$. Let $p_g(S) := \dim H^0(S, \Omega^2_S)$ (geometric genus of S). If $p_g(S) \neq 0$, then $T(S) \neq 0$ and in fact $T(S)$ is "infinite dimensional".*

We shall below make this notion more precise but it implies that the "size" of $T(S)$ is so large that it can *not* be parametrized by an algebraic variety.

Write shortly $\mathrm{CH}_0(S)_0 := \mathrm{CH}^{alg}_0(S)$.

It is somewhat more convenient to formulate things in terms of $\mathrm{CH}_0(S)_0$ itself but remember that the "size" of $T(S)$ and $\mathrm{CH}_0(S)_0$ only differ by a finite number namely the dimension of $\mathrm{Alb}(S)$ (which itself is half the dimension of $H^{2d-1}(S)$).

Consider more generally $X = X_d$ and its $n-$fold *symmetric product* $X^{(n)}$ (the quotient of the $n-$fold product $X \times X \times \cdots \times X$ by the symmetric group;

$X^{(n)}$ has mild singularities!). Clearly $X^{(n)}$ parametrizes the $0-$cycles of degree n and we have a map

$$\begin{aligned} \varphi_n : X^{(n)} \times X^{(n)} &\longrightarrow \mathrm{CH}_0(X)_0 \\ (Z_1, Z_2) &\longmapsto \text{class of } (Z_1 - Z_2). \end{aligned}$$

Mumford calls $\mathrm{CH}_0(X)_0$ *finite dimensional* if there exists an n such that φ_n is surjective; otherwise *infinite dimensional*.

Remark 5.2. One can elaborate further on this concept as follows. There is the following fact (see for instance [V2, 10.1]): the fibers of φ_n consist of a countable number of algebraic varieties (this is proved via the existence of the Chow varieties or via the existence of the Hilbert schemes) Since each of these subvarieties has a bounded dimension (at most $2nd$) it makes sense to take the maximum and call that the dimension of the fiber. Let r_n be the dimension of the generic fiber and consider $2nd - r_n$; this can be considered as the "dimension" of $\mathrm{Im}\varphi_n$. So intuitively "$\dim \mathrm{CH}_0(X)_0 = \varinjlim_n (2nd - r_n)$". Now there is the following fact (see [V2, 10.10]: $\mathrm{CH}_0(X)_0$ is finite dimensional if and only if $\varinjlim_n (2nd - r_n)$ is finite.

We mention (in passing) the following beautiful *theorem of Roitman* (see [V2, Prop. 10.11]):

Theorem 5.3 (Roitman, 1972)**.** *If $\mathrm{CH}_0(X)_0$ is finite dimensional then the albanese morphism $\alpha_X : \mathrm{CH}_0(X)_0 \twoheadrightarrow \mathrm{Alb}(X)$ is an isomorphism.*

Proof. We refer to [V2, 10.1.2]. □

In this lecture we want to reduce the proof of the Mumford's theorem to the method used by Bloch (see below). For that we need one further result for which we must refer to Voisin's book ([V2, 10.12]).

Lemma 5.4. *The following properties are equivalent:*

a. $\mathrm{CH}_0(X)_0$ *is finite dimensional.*
b. *If $C = X \cap F_1 \cap \cdots \cap F_{d-1} \overset{j}{\hookrightarrow} X$ is a smooth curve cut out on X by hypersurfaces F_i then the induced homomorphism $j_* : J(C) = \mathrm{CH}^0(C)_0 \to \mathrm{CH}^0(X)_0$ is surjective.*

Remark 5.5. For a very precise list of properties equivalent to finite dimensionality one can look to proposition 1.6 of the paper by Jannsen [J1] in the proceedings of the Seattle Conference on motives (1991).

5.2. **Reformulation and generalization by Bloch.**

5.2.1. *Reformulation of finite dimensionality.* S. Bloch introduced in 1976 the notion of "*weak representability*".

Let $\Omega \supset k$ be a so-called "*universal domain*", i.e. Ω is an algebraically closed field of infinite transcendence degree over k; hence every $L \supset k$ of finite transcendence degree over k can be embedded in Ω, i.e. $k \subset L \subset \Omega$ (for example, if $k = \bar{\mathbb{Q}}$ then one can take $\Omega = \mathbb{C}$).

Definition 5.6. Let X be defined over k. $\mathrm{CH}^j_{alg}(X)$ is *weakly representable* if there exists a curve C smooth, but not necessarily irreducible, and a cycle class $T \in \mathrm{CH}^j(C \times X)$ such that the corresponding homomorphism T_* : $\mathrm{CH}_0^{alg}(C_L) \to \mathrm{CH}^j_{alg}(X_L)$ is surjective for all L with $k \subset L \subset \Omega$ and $L = \bar{L}$.

Remark 5.7.

a. If $T(S) = 0$ then it is easy to see that $\mathrm{CH}_0(S)_0$ is weakly representable. Namely $T(S) = 0$ gives $\mathrm{CH}_0(S)_0 \xrightarrow{\sim} \mathrm{Alb}(S)$ and by taking a sufficiently general curve C on S we get a surjective map $J(C) \twoheadrightarrow \mathrm{Alb}(S)$.

b. Assume tacitly that we have chosen on each connected component of C a "base point" e such that T_* is defined as $T(x) - T(e)$ for $x \in C$.

c. If follows (easily) from the lemma 5.4 mentioned in section 5.1 above that $\mathrm{CH}_0(X)_0$ is finite dimensional if and only if it is weakly representable.

d. Often this notion is denoted by "representability", however it seems better to use the name "weak representability" to distinguish it from the much stronger concept of representability in the sense of Grothendieck.

5.2.2. *Transcendental cohomology.* Let again $X = X_d$ be as usual and assume we have choosen a Weil cohomology theory with coefficient field $F \supset \mathbb{Q}$. Consider the cycle map

$$\mathrm{NS}(X) \otimes_{\mathbb{Q}} F \longrightarrow H^2(X)$$

(recall the definition $\mathrm{NS}(X) = \mathrm{CH}^1(X)/\mathrm{CH}^1_{alg}(X)$). Put $H^2(X)_{alg}$ for the image and $H^2(X)_{tr} := H^2(X)/H^2(X)_{alg}$; $H^2(X)_{tr}$ is called the group of *trascendental (cohomology) cycles.*

So if $X = S$ a surface then we have via Poincaré duality an orthogonal decomposition

$$H^2(S) = H^2(S)_{alg} \oplus H^2(S)_{tr}.$$

5.2.3. *A theorem by Bloch.*

Theorem 5.8 (Bloch 1979)**.** *(See* [B, p. I. 24]*) Let S be an algebraic surface defined over $k = \bar{k}$. Assume that $H^2(S)_{tr} \neq 0$, then $\mathrm{CH}^2_{alg}(S)$ is not weakly representable.*

The proof will be given in section 5.3 below. Bloch's theorem implies Mumford's theorem. Namely let $k = \mathbb{C}$ and $p_g(S) \neq 0$. Now $p_g(S) = \dim H^0(S, \Omega^2) = \dim H^{2,0}(S)$ and since $H^2(S)_{alg} \subset H^{1,1}(S)$ we have $H^2(S)_{tr} \supset H^{20}(S)$. Therefore $p_g(S) \neq 0$ implies $H^2(S)_{tr} \neq 0$, hence $\mathrm{CH}^2_{alg}(S)$ is not weakly representable hence $\mathrm{CH}^2_{alg}(S)$ is not finite dimensional and in particular $T(S) \neq 0$.

5.3. **A result on the diagonal.** First as a matter of notation if $Z \in \mathcal{Z}^i(X)$ let us write $|Z| \subset X$ for the support of Z.

Theorem 5.9 (Bloch 1979, Bloch-Srinivas 1983)**.** *Let $X = X_d$ be smooth, irreducible, projective and defined over k. Let Ω be a "universal domain" as before. Assume that there exists (over k) a closed algebraic subset $Y \subsetneq X$ such that for $U = X - Y$ we have $\mathrm{CH}_0(U_\Omega) = 0$. Then there exists d-dimensional cycles Γ_1 and Γ_2 with supports $|\Gamma_1| \subset X \times Y$ and $|\Gamma_2| \subset W \times X$ where $W \subsetneq X$ is a closed algebraic subset (defined over k) and an integer $N > 0$ such that*

$$N \cdot \Delta(X) = \Gamma_1 + \Gamma_2$$

where $\Delta(X) \subset X \times X$ is the diagonal.

Proof. Take the generic point η of X. Consider the $0-$cycle (η) in $\mathrm{CH}_0(X_L)$ where $L = k(\eta)$, in fact $(\eta) \in \mathrm{CH}_0(U_L)$.

Lemma 5.10. $\ker(\mathrm{CH}_0(U_L) \to \mathrm{CH}_0(U_\Omega))$ *is torsion.*

Proof. We proceed in three steps: $L \subset L' \subset \bar{L} \subset \Omega$ with $[L' : L]$ finite. By the first step $L' \supset L$ the kernel is finite because of the existence of a norm map $\mathrm{CH}_0(U_{L'}) \to \mathrm{CH}_0(U_L)$ and the composition $\mathrm{CH}(U_L) \to \mathrm{CH}(U_{L'}) \to \mathrm{CH}(U_L)$ is $[L' : L]$id. Next the step $L \subset \bar{L}$ is also torsion since $\mathrm{CH}_0(U_{\bar{L}}) = \varinjlim \mathrm{CH}_0(U_{L'})$. Finally the step $\mathrm{CH}_0(U_{\bar{L}}) \to \mathrm{CH}_0(U_\Omega)$ is an isomorphism because $\Omega = \varinjlim R$ where R are finitely generated $\bar{L}-$algebras and $\mathrm{CH}(U_{\bar{L}}) \to \mathrm{CH}(U \times_{\bar{L}} \mathrm{Spec} R)$ is an isomorphism because we get a section by taking a $\bar{L}-$rational point in $\mathrm{Spec}(R)$. This proves the lemma. □

Returning to the $0-$cycle $(\eta) \in \mathrm{CH}_0(U_L)$ we see from our assumption that $\mathrm{CH}_0(U_\Omega) = 0$ and from the lemma that there exists $N > 0$ such that $N \cdot (\eta) = 0$ in $\mathrm{CH}_0(U_L)$. Now we apply to X_L the localization theorem 1.20 from lecture I and we see that in $\mathrm{CH}_0(Y_L)$ there exists a $0-$cycle A such that in $\mathrm{CH}_0(X_L)$ the $0-$cycle

$$N \cdot (\eta) - A = 0.$$

Remark 5.11. Strictly speaking we assumed in lecture I that the base field is algebraically closed; however the localization sequence holds for any base field ([F, Prop. 1.8]) so we can apply it also in our case to $L = k(\eta)$.

Next we consider X_L as the fibre in $X \times X$ over the point η (η in the first factor, X_η in the second) and take in $X \times X$ the $k-$Zariski closure of $N \cdot (\eta) - A$. Then we get a d-dimensional cycle $(N \cdot \Delta(X) - \Gamma_1)$ on $X \times X$ where $\Gamma_1 \in \mathcal{Z}_d(X \times X)$ and $|\Gamma_1| \subset X \times Y$, Γ_1 restricted to $X_{k(\eta)}$ is A and the restriction of the cycle $N\Delta(X) - A$ to $\mathrm{CH}_0(X_{k(\eta)}$ is zero. However

$$\mathrm{CH}_0(X_{k(\eta)}) = \varinjlim_D \mathrm{CH}_0((X - D) \times X)$$

where the limit runs over divisors $D \subset X$. Therefore there exists a divisor, say $D = W$, and a cycle $\Gamma_2 \in \mathcal{Z}_d(X \times X)$ with $|\Gamma_2| \subset W \times X$ such that $N \cdot \Delta(X) = \Gamma_1 + \Gamma_2$. This completes the proof. □

5.3.1. *End of the proof of Mumford's theorem.*

Claim 5.12. *Theorem 5.9 implies theorem 5.8 (and we have already seen in section 5.2.3 that theorem 5.8 implies Mumford's theorem).*

Proof of the claim. Let $X = S$ be a surface such that $H^2(S)_{tr} \neq 0$. Now if $\mathrm{CH}^2_{alg}(S) = \mathrm{CH}_0(S)_0$ is weakly representable then there exists a curve C and $T \in \mathrm{CH}^2(C \times S)$ such that $\mathrm{CH}_0^{alg}(C_L) \to \mathrm{CH}_0^{alg}(S_L)$ is surjective for all $k \subset L \subset \Omega$, $L = \bar{L}$. Therefore if we take $Y = \mathrm{pr}_2 T$ then the condition of the theorem of Bloch-Srinivas is satisfied. Now we have lemma 5.13 below and clearly this proves the claim because $N \cdot \Delta(X)$ operates non-trivially on $H^2(S)_{tr}$, since clearly it operates as multiplication by N. $\square$

Lemma 5.13. *Both correspondences Γ_1 and Γ_2 operate trivially on $H^2(S)_{tr}$.*

Proof. Let us start with Γ_2. Now $|\Gamma_2| \subset C \times S$ where $\iota : C \hookrightarrow S$ is a curve. We have a commutative diagram

$$\begin{tikzcd} C \times S \arrow[r, "\iota_1"] \arrow[d, "q=\mathrm{pr}_1"'] & S \times S \arrow[d, "\mathrm{pr}_1=p"] \\ C \arrow[r, "\iota"] & S \end{tikzcd}$$

where $\iota_1 = \iota \times \mathrm{id}_S$. Now let $\alpha \in H^2(S)_{tr}$. We have $\Gamma_2(\alpha) = (\mathrm{pr}_2)_*\{p^*(\alpha) \cup (\iota_1)_*(\Gamma_2)\}$. By the projection formula $p^*(\alpha) \cup (\iota_1)_*(\Gamma_2) = (\iota_1)_*\{\iota_1^* p^*(\alpha) \cup \Gamma_2\}$; now $\iota_1^* p^*(\alpha) = q^*\iota^*(\alpha)$, but $\iota^*(\alpha) = 0$ because $\alpha \in H^2(S)_{tr}$ is orthogonal to the class of C in $H^2(S)$. Therefore $\Gamma_2(\alpha) = 0$.

Next Γ_1. We have $|\Gamma_1| \subset S \times C'$ where again $\iota' : C' \to S$ is a curve. We have a commutative diagram

$$\begin{tikzcd} S \times C' \arrow[r, "\iota_2"] \arrow[d, "q'=\mathrm{pr}_2"'] & S \times S \arrow[d, "\mathrm{pr}_2=p'"] \\ C' \arrow[r, "\iota'"] & S \end{tikzcd}$$

where $\iota_2 = \mathrm{id}_S \times \iota'$. For $\alpha \in H^2(S)_{tr}$ we get $\Gamma_1(\alpha) = (\mathrm{pr}_2)_*\{(\iota_2)_*(\Gamma_1) \cup \mathrm{pr}_1^*(\alpha)\}$, by the projection formula $(\iota_2)_*(\Gamma_1) \cup \mathrm{pr}_1^*(\alpha) = (\iota_2)_*\{\Gamma_1 \cup \iota_2^* \mathrm{pr}_1^*(\alpha)\}$. Writing $\beta = \Gamma_1 \cup \iota_2^* \mathrm{pr}_1^*(\alpha)$ we get $\Gamma_1(\alpha) = (\mathrm{pr}_2)_*(\iota_2)_*(\beta) = (\iota')_* q'_*(\beta) \in H^2(S)_{alg} = \mathrm{NS}(S)$, therefore the image in $H^2(S)_{tr}$ is zero. $\square$

5.3.2. *Bloch's conjecture.* Let S be a surface as before defined over $k = \bar{k}$. Assume $H^2(S)_{tr} = 0$. Then Bloch conjectures that the Albanese kernel $T(S) = 0$, i.e. that the albanese map $\alpha_S : \mathrm{CH}^2_{alg}(S) \to \mathrm{Alb}(S)$ is an isomorphism, which implies that $\mathrm{CH}^2_{alg}(S)$ is "finite dimensional".

Bloch's conjecture has been proved for all surfaces which are *not* of "general type" [V2, 11.10], i.e. surfaces for which the so-called "Kodaira dimension" is less than two. For surfaces of general type it has only been proved in special cases, for instance for so-called Godeaux surfaces.

Bloch's conjecture is one of the most important conjectures in the theory of algebraic cycles.

5.4. **References for lecture V.** For Mumford's theorem, see [V2, Chap. 10]. For Bloch's theorem see Bloch's book [B] on his Duke lectures held in 1979; there is now a second edition which appeared in Cambridge Univ. Press. For the Bloch-Srinivas theorem and its consequences see the original paper [B-S]. One can find this material also in Chapters 10 and 11 of [V2].

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