THE HILALI CONJECTURE FOR HYPERELLIPTIC SPACES

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Abstract. Hilali Conjecture predicts that for a simply-connected elliptic space, the total dimension of the rational homotopy does not exceed that of the rational homology. Here we give a proof of this conjecture for a class of elliptic spaces known as hyperelliptic.

1. Introduction

Let X be a simply-connected CW-complex. Then X is said to be of elliptic type if both $\dim H^*(X,\mathbb{Q}) < \infty$ and $\dim \pi_*(X)\otimes\mathbb{Q} < \infty$. For these spaces, Hilali conjetured in [5] the following:

Conjecture 1.1. *If X is a simply-connected CW-complex of elliptic type, then*

$$\dim \pi_*(X)\otimes\mathbb{Q} \leq \dim H^*(X,\mathbb{Q})\,.$$

By the theory of minimal models of Sullivan [3], the rational homotopy type of X is encoded in a differential algebra (A,d) called *the minimal model* of X. This is a free graded algebra $A=\Lambda V$, generated by a graded vector space $V=\bigoplus_{k\geq 2} V^k$, and with decomposable differential, i.e., $d: V^k \to (\Lambda^{\geq 2}V)^{k+1}$. It satisfies that:

$$\begin{aligned} V^k &= (\pi_k(X)\otimes\mathbb{Q})^*\,, \\ H^k(\Lambda V,d) &= H^k(X,\mathbb{Q})\,. \end{aligned}$$

Therefore the Hilali conjecture can be rewritten as follows: for a finite-dimensional graded vector space V (in degrees bigger or equal than two), we have

$$\dim V \leq \dim H^*(\Lambda V,d)$$

for any decomposable differential d on ΛV.

An elliptic space X is called of pure type if its minimal model $(\Lambda V,d)$ satisfies that $V=V^{even}\oplus V^{odd}$, $d(V^{even})=0$ and $d(V^{odd})\subset \Lambda V^{even}$. Also X is called hyperelliptic if $d(V^{even})=0$ and $d(V^{odd})\subset \Lambda^+V^{even}\otimes\Lambda V^{odd}$.

In his thesis [5] in 1990, Hilali proved Conjecture 1.1 for elliptic spaces of pure type. The conjecture is known to hold [6, 7] also in several cases: H-spaces, nilmanifolds, symplectic and cosymplectic manifolds, coformal spaces with only odd-degree generators, and formal spaces. Hilali and Mamouni [6, 7] have also proved Conjecture 1.1 for hyperelliptic spaces under various conditions in the homotopical and homological Euler characteristics.

The main result of this paper is the following:

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Theorem 1.2. *Conjecture 1.1 holds for hyperelliptic spaces.*

We shall start by proving it for elliptic spaces of pure type in section 3. This requires to reduce the question to a problem about Tor functors of certain modules of finite length over a polynomial ring. We solve it by using a semicontinuity result for the Tor functor. Then in section 4 we prove theorem 1.2 for hyperelliptic spaces. For this we have to prove a semicontinuity result for the homology of elliptic spaces, and apply it to reduce the general case to the case in which the minimal model only has generators of odd degree and zero differential. We give two different proofs of an inequality from which the result follows.

2. Minimal models

We recall some definitions and results about minimal models [2]. Let (A, d) be a *differential algebra*, that is, A is a (positively) graded commutative algebra over the rational numbers, with a differential d which is a derivation, i.e., $d(a \cdot b) = (da) \cdot b + (-1)^{\deg(a)} a \cdot (db)$, where $\deg(a)$ is the degree of a. We say that A is connected if $A^0 = \mathbb{Q}$, and simply-connected if moreover $A^1 = 0$.

A simply-connected differential algebra (A, d) is said to be *minimal* if:

(1) A is free as an algebra, that is, A is the free algebra ΛV over a graded vector space $V = \oplus_{k \geq 2} V^k$, and
(2) For $x \in V^k$, $dx \in (\Lambda V)^{k+1}$ has no linear term, i.e., it lives in $\Lambda V^{>0} \cdot \Lambda V^{>0} \subset \Lambda V$.

Let (A, d) be a simply-connected differential algebra. A minimal model for (A, d) is a minimal algebra $(\Lambda V, d)$ together with a quasi-isomorphism $\rho : (\Lambda V, d) \to (A, d)$ (that is, a map of differential algebras such that $\rho_* : H^*(\Lambda V, d) \to H^*(A, d)$ is an isomorphism). A minimal model for (A, d) exists and it is unique up to isomorphism.

Now consider a simply-connected CW-complex X. There is an algebra of piecewise polynomial rational differential forms $(\Omega^*_{PL}(X), d)$ defined in [3, Chap. VIII]. A minimal model of X is a minimal model $(\Lambda V_X, d)$ for $(\Omega^*_{PL}(X), d)$. We have that

$$\begin{aligned} V^k &= (\pi_k(X) \otimes \mathbb{Q})^* , \\ H^k(\Lambda V, d) &= H^k(X, \mathbb{Q}) . \end{aligned}$$

A space X is elliptic [1] if both $\sum \dim \pi_k(X) \otimes \mathbb{Q} < \infty$ and $\sum \dim H^k(X, \mathbb{Q}) < \infty$. Equivalently, if $(\Lambda V, d)$ is the minimal model, we require that both V and $H^*(\Lambda V, d)$ are finite dimensional. For elliptic spaces, the Euler-Poincaré and the homotopic characteristics are well defined:

$$\chi = \sum_{i \geq 0} (-1)^i \dim H^i(\Lambda V, \mathbb{Q}),$$
$$\chi_\pi = \sum_{i \geq 0} (-1)^i \dim \pi_i(X) \otimes \mathbb{Q} = \dim V^{even} - \dim V^{odd}.$$

We refer the reader to [2, Thm. 32.10] for the proof of the following:

Proposition 2.1. *Let $(\Lambda V, d)$ be an elliptic minimal model. Then $\chi \geq 0$ and $\chi_\pi \leq 0$. Moreover, $\chi_\pi < 0$ if and only if $\chi = 0$.*

In his thesis [5], M. Hilali conjectured that for elliptic spaces:

$$\dim \pi_*(X) \otimes \mathbb{Q} \leq \dim H^*(X, \mathbb{Q})\,.$$

In algebraic terms, this is equivalent to

$$\dim V \leq \dim H^*(\Lambda V, d)\,,$$

whenever $(\Lambda V, d)$ is a minimal model with $\dim V < \infty$. Note that finiteness of both $\dim H^*(X, \mathbb{Q})$ and $\dim \pi_*(X) \otimes \mathbb{Q}$ is necessary. Otherwise, one can easily construct counterexamples such as $X = S^3 \vee S^3$.

3. Proof of the Hilali conjecture for elliptic spaces of pure type

A minimal model $(\Lambda V, d)$ is of *pure type* if $V = V^{even} \oplus V^{odd}$, with

$$d(V^{even}) = 0, \quad d(V^{odd}) \subset \Lambda V^{even}.$$

An elliptic space is of pure type if its minimal model is so. These spaces are widely studied in [2, §32]. By proposition 2.1, we have that $\dim V^{even} - \dim V^{odd} \leq 0$. Let $n = \dim V^{even}$ and $n + r = \dim V^{odd}$, where $r \geq 0$. Write $x_1, \ldots, x_n$ for the generators of even degree, and $y_1, \ldots, y_{n+r}$ for the generators of odd degree. Then $dx_i = 0$, and $dy_j = P_j(x_1, \ldots, x_n)$, where P_j are polynomials without linear terms.

In this section we prove the following:

Theorem 3.1. *The Hilali conjecture holds for elliptic spaces of pure type.*

3.1. **Expressing the homology as a Tor functor.** To work over nice modules we would like to reorder the generators $y_1, \ldots, y_{n+r}$, so that $P_1, \ldots, P_n$ form a regular sequence in $\Lambda(x_1, \ldots, x_n)$. Recall that this means that the image of P_i in $\Lambda(x_1, \ldots, x_n)/(P_1, \ldots, P_{i-1})$ is not a zero divisor, for any $i = 1, \ldots, n$. But this is not possible in general, as shown by the following example.

Example 3.2. *Let* $V = \mathbb{Q}\langle x_1, x_2, y_1, y_2, y_3\rangle$, *where* $\deg(x_1) = 2$ *and* $\deg(x_2) = 6$. *Define a differential* d *on* ΛV *by*

$$dy_1 = x_1^6 + x_2^2, \quad dy_2 = x_1^9 + x_2^3, \quad dy_3 = x_1^4 x_2 + x_1 x_2^2.$$

Then $(\Lambda V, d)$ *is a pure minimal model. It can be proved that is elliptic if and only if there exist exact powers of* x_1 *and* x_2. *This is the case, since* $2x_1^{10} = d(x_1^4 y_1 + x_1 y_2 - x_2 y_3)$ *and* $2x_2^4 = d(x_2^2 y_1 + x_2 y_2 - x_1^5 y_3)$. *But for the same reason, models* $(\Lambda(x_1, x_2, y_i, y_j), d)$ *are not elliptic for any choice of indices* i, j. *This amounts to say that* dy_i, dy_j *are not a regular sequence in* $\Lambda(x_1, x_2)$.

However, Halperin showed in [4, Lemma 8] that pure models always admit a basis $z_1, \ldots, z_{n+r}$ of V^{odd} such that $dz_1, \ldots, dz_n$ is a regular sequence in $\Lambda(x_1, \ldots, x_n)$. This basis is not necessarily homogeneous but it is possible to preserve the lower grading induced by the number of odd elements, that is

$$(\Lambda V)_q^p = (\Lambda V^{even} \otimes \Lambda^q V^{odd})^p.$$

This grading passes to cohomology and by taking into account the quasi-isomorphisms

$$(\Lambda(x_1, \ldots, x_n, y_1, \ldots, y_{n+r}),\ d) \xrightarrow{\sim} (\Lambda(x_1, \ldots, x_n, z_1, \ldots, z_{n+r}),\ d)$$
$$(\Lambda(x_1, \ldots, x_n, z_1, \ldots, z_n),\ d) \xrightarrow{\sim} (\Lambda(x_1, \ldots, x_n)/(dz_1, \ldots, dz_n),\ d)$$

with respect to the lower grading, one deduces that:

$$H_*(\Lambda V, d) \cong H_*(\Lambda(x_1, \dots, x_n)/(dz_1, \dots, dz_n) \otimes \Lambda(z_{n+1}, \dots, z_{n+r}), d).$$

So let $z_1, \dots, z_{n+r}$ be a basis such that $dz_1, \dots, dz_n$ form a regular sequence. Put $P_j = dz_j$ for $j = 1, \dots, n+r$ and consider the module

$$M = \mathbb{Q}[x_1, \dots, x_n]/(P_1, \dots, P_n)$$

over the ring

$$R = \mathbb{Q}[x_1, \dots, x_n]\,.$$

Consider the ring

$$S = \mathbb{Q}[\lambda_1, \dots, \lambda_r]$$

and the map $f : S \to R$, $\lambda_i \mapsto P_{n+i}$. Then M becomes an S-module.

Consider also the S-module

$$\mathbb{Q}_0 = S/(\lambda_1, \dots, \lambda_r).$$

Then we have the following:

Proposition 3.3. $H_*(\Lambda V, d) \cong \mathrm{Tor}^*_S(M, \mathbb{Q}_0)$.

Proof. Let $U = \langle z_1, \dots, z_n\rangle$, $W = \langle z_{n+1}, \dots, z_{n+r}\rangle$ so that $V^{odd} = U \oplus W$. Then the map $(\Lambda V^{even} \oplus U, d) \to (M, 0)$ is a quasi-isomorphism. Actually, the Koszul complex

$$R \otimes \Lambda^n U \to R \otimes \Lambda^{n-1} U \to \dots \to R \otimes \Lambda^1 U \to R \to M$$

is exact, which means that $(R \otimes \Lambda U, d) \xrightarrow{\sim} (M, 0)$.

Therefore

$$(\Lambda V, d) = (R \otimes \Lambda U \otimes \Lambda W, d) \xrightarrow{\sim} (M \otimes \Lambda W, d')\,, \tag{1}$$

is an isomorphism, where the differential d' is defined as zero on M, and $d' z_{n+i} = \bar{P}_{n+i} \in M$. This can be seen as follows: the map (1) is a map of differential algebras. Grading both algebras in such a way that $\Lambda^k W$ has degree k, we get two spectral sequences. The map between their E_1-terms is

$$H^*(R \otimes \Lambda U, d) \otimes \Lambda W \to M \otimes \Lambda W\,.$$

As this is an isomorphism, it follows that the map in the E_∞-terms is also an isomorphism. The E_∞-terms are the homology of both algebras in (1). So the map (1) is an isomorphism.

Finally, we have to identify $H^*(M \otimes \Lambda W, d') \cong \mathrm{Tor}^*_S(M, \mathbb{Q}_0)$. Note that the homology of $(M \otimes \Lambda W, d')$ is computed as follows: take the Koszul complex

$$S \otimes \Lambda^r W \to S \otimes \Lambda^{r-1} W \to \dots \to S \otimes \Lambda^1 W \to S \to \mathbb{Q}_0\,,$$

and tensor it with M over S (with the S-module structure given above), to get

$$(M \otimes_S (S \otimes \Lambda W), d') = (M \otimes \Lambda W, d').$$

The homology of this computes $\mathrm{Tor}^*_S(M, \mathbb{Q}_0)$. □

Lemma 3.4. *Under our assumptions,*

$$\dim \mathrm{Tor}^0_S(M, \mathbb{Q}_0) \geq n+1 \quad \textit{and} \quad \dim \mathrm{Tor}^r_S(M, \mathbb{Q}_0) \geq n+1.$$

Proof. Clearly,

$$\mathrm{Tor}^0_S(M,\mathbb{Q}_0) = M\otimes_S \mathbb{Q}_0 = M/(\bar P_{n+1},\dots,\bar P_{n+r}) = R/(P_1,\dots,P_{n+r})\,.$$

As all the polynomials $P_1,\dots,P_{n+r}$ have no linear part, this module contains the constant and linear monomials at least, so $\dim \mathrm{Tor}^0_S(M,\mathbb{Q}_0)\geq n+1$.

For the other inequality, note that $\mathrm{Tor}^r_S(M,\mathbb{Q}_0)$ is the kernel of $M\otimes\Lambda^r W\to M\otimes \Lambda^{r-1}W$, i.e., the kernel of

$$(P_{n+1},\dots,P_{n+r}) : M\to M\oplus \overset{(r)}{\cdots} \oplus M\,. \tag{2}$$

Now we use the following fact: as M is a complete intersection R-module (it is the quotient of R by a regular sequence), it has Poincaré duality in the sense that there is a map $M\to\mathbb{Q}$ such that $\Gamma: M\otimes M \overset{\text{mult}}{\longrightarrow} M\to \mathbb{Q}$ is a perfect pairing. Take elements $\nu,\mu_j\in M$, $j=1,\dots,n$, such that

$$\begin{array}{ll} \Gamma(\nu,x_j)=0,\ j=1,\dots,n, & \Gamma(\nu,1)=1,\\ \Gamma(\mu_j,x_k)=\delta_{jk},\ j,k=1,\dots,n, & \Gamma(\mu_j,1)=0,\\ \Gamma(\nu,Q)=\Gamma(\mu_j,Q)=0, \quad \text{for any quadratic } Q\in R. & \end{array}$$

Since the elements ν,μ_j are in the kernel of (2) and they are linearly independent, we get $\dim\mathrm{Tor}^r_S(M,\mathbb{Q}_0)\geq n+1$. □

3.2. **Semicontinuity theorem.** We are going to prove a semicontinuity theorem for the Tor functors $\mathrm{Tor}^k_S(M,\mathbb{Q}_0)$ for flat families of modules M of finite length (i.e., finite-dimensional as $\mathbb{Q}$-vector spaces).

Consider a variable t. A family of S-modules is a module $\mathcal{M}$ over $S[t]$ such that for each t_0, the S-module

$$M_{t_0} = \mathcal{M}/(t-t_0)$$

is of finite length. We say that $\mathcal{M}$ is flat over $\mathbb{Q}[t]$ if it is a flat $\mathbb{Q}[t]$-module, under the inclusion $\mathbb{Q}[t]\hookrightarrow S[t]$. Consider $\mathcal{M}$ as a $\mathbb{Q}[t]$-module. Then

$$\mathcal{M}\cong \mathbb{Q}[t]^N\oplus \frac{\mathbb{Q}[t]}{(t-t_1)^{b_1}}\oplus\dots\oplus\frac{\mathbb{Q}[t]}{(t-t_l)^{b_l}}\,,$$

for some $N\geq 0$, $l\geq 0$, $1\leq b_1\leq\dots\leq b_l$. The module is flat if and only if there is no torsion part, i.e., $l=0$ (to see this, tensor the exact sequence $0\to\mathbb{Q}[t]\overset{t-t_i}{\longrightarrow}\mathbb{Q}[t]\to\mathbb{Q}[t]/(t-t_i)\to 0$ with $\mathcal{M}$). Note that for generic ξ, $\mathrm{length}(M_\xi)=N$. Therefore the flatness is equivalent to $\mathcal{M}/(t-t_i)$ being of length N, i.e.,

$$\mathcal{M} \text{ is flat } \iff \mathrm{length}(M_t)=N,\ \forall t\,.$$

Lemma 3.5. *For any flat family* $\mathcal{M}$,

$$\dim\mathrm{Tor}^k_S(M_0,\mathbb{Q}_0)\geq \dim\mathrm{Tor}^k_S(M_\xi,\mathbb{Q}_0),$$

for generic $\xi\in\mathbb{Q}$.

Proof. Let us resolve $\mathcal{M}$ as a $S[t]$-module:

$$0\to S[t]^{a_r}\to\dots\to S[t]^{a_0}\to\mathcal{M}\to 0\,. \tag{3}$$

As $\mathcal{M}$ is flat as $\mathbb{Q}[t]$-module, if we tensor the inclusion $\mathbb{Q}[t] \stackrel{t}{\hookrightarrow} \mathbb{Q}[t]$ by $\mathcal{M}$ over $\mathbb{Q}[t]$, we have that $\mathcal{M} \stackrel{t}{\hookrightarrow} \mathcal{M}$ is an inclusion. Hence the sequence

$$0 \to \mathcal{M} \stackrel{t}{\hookrightarrow} \mathcal{M} \to \mathcal{M}/(t) \to 0$$

is exact. But this sequence is the sequence $0 \to S[t] \to S[t] \to S[t]/(t) \to 0$ tensored by $\mathcal{M}$ over $S[t]$. Hence $\mathrm{Tor}^1_{S[t]}(\mathcal{M}, S[t]/(t)) = 0$. Obviously $\mathrm{Tor}^j_{S[t]}(\mathcal{M}, S[t]/(t)) = 0$ for $j \geq 2$ (since the resolution $S[t]/(t)$ has two terms).

Using the above, we can tensor $(3)\otimes_{S[t]} S[t]/(t)$ to get an exact sequence:

$$0 \to S^{a_r} \to \ldots \to S^{a_0} \to M_0 \to 0\,. \tag{4}$$

Now we tensor (4) by $\otimes_S \mathbb{Q}_0$ and take homology to obtain $\mathrm{Tor}^*_S(M_0, \mathbb{Q}_0)$. But

$$(4) \otimes_S \mathbb{Q}_0 = (3) \otimes_{S[t]} \mathbb{Q}_0 = ((3) \otimes_{S[t]} \mathbb{Q}[t]) \otimes_{\mathbb{Q}[t]} \mathbb{Q}[t]/(t) = (5) \otimes_{\mathbb{Q}[t]} \mathbb{Q}[t]/(t)\,,$$

where $\mathbb{Q}_0 = S[t]/(\lambda_1, \ldots, \lambda_r, t)$, and

$$0 \to \mathbb{Q}[t]^{a_r} \to \ldots \to \mathbb{Q}[t]^{a_0} \to \mathcal{F} = \mathcal{M}/(\lambda_1, \ldots, \lambda_r) \to 0. \tag{5}$$

(This is just a complex, maybe not exact.) Analogously,

$$\mathrm{Tor}^*_S(M_0, \mathbb{Q}_\xi) = H^*((5) \otimes_{\mathbb{Q}[t]} \mathbb{Q}[t]/(t-\xi))\,.$$

So it remains to see that for a complex $L_\bullet$ of free $\mathbb{Q}[t]$-modules like (5), it holds that

$$\dim H^k(L_\bullet \otimes \mathbb{Q}[t]/(t-\xi)) \leq \dim H^k(L_\bullet \otimes \mathbb{Q}[t]/(t)),$$

for generic ξ. (Tensor products are over $\mathbb{Q}[t]$, which we omit in the notation henceforth.) For proving this, just split (5) as short exact sequences

$$0 \to Z_i \to L_i \to B_{i-1} \to 0, \tag{6}$$

and note that Z_i, B_i are free $\mathbb{Q}[t]$-modules, being submodules of free modules. So $Z_i = \mathbb{Q}[t]^{z_i}$ and $B_i = \mathbb{Q}[t]^{b_i}$. Now $0 \to B_i \to Z_i \to H^i(L_\bullet) \to 0$ gives that

$$H^i(L_\bullet) = \mathbb{Q}[t]^{z_i - b_i} \oplus \text{torsion}.$$

For generic ξ, we have $\dim H^i(L_\bullet \otimes \mathbb{Q}[t]/(t-\xi)) = z_i - b_i$. Hence

$$\begin{array}{ccccccccc} 0 & \to & Z_i \otimes \mathbb{Q}[t]/(t) & \to & L_i \otimes \mathbb{Q}[t]/(t) & \to & B_{i-1} \otimes \mathbb{Q}[t]/(t) & \to & 0 \\ & & \downarrow & & || & & \downarrow & & \\ 0 & \to & Z_i(L_\bullet \otimes \mathbb{Q}[t]/(t)) & \to & L_i \otimes \mathbb{Q}[t]/(t) & \to & B_{i-1}(L_\bullet \otimes \mathbb{Q}[t]/(t)) & \to & 0\,. \end{array}$$

The first sequence is (6) tensored by $\mathbb{Q}[t]/(t)$. Thus the last vertical map is surjective, and the first vertical map is injective.

Therefore, we get:

$$\begin{aligned} \dim H^i(L_\bullet \otimes \mathbb{Q}[t]/(t)) &= \dim Z_i(L_\bullet \otimes \mathbb{Q}[t]/(t)) - \dim B_i(L_\bullet \otimes \mathbb{Q}[t]/(t)) \\ &\geq \dim Z_i \otimes \mathbb{Q}[t]/(t) - \dim B_i \otimes \mathbb{Q}[t]/(t) \\ &= \dim H^i(L_\bullet) \otimes \mathbb{Q}[t]/(t) - \dim \mathrm{Tor}_1^{\mathbb{Q}[t]}(H^i(L_\bullet), \mathbb{Q}[t]/(t)) \\ &= z_i - b_i\,, \end{aligned}$$

where we have used in the third line that there is an exact sequence

$$0 \to \mathrm{Tor}_1^{\mathbb{Q}[t]}(H^i(L_\bullet), \mathbb{Q}[t]/(t)) \to B_i \otimes \mathbb{Q}[t]/(t) \to Z_i \otimes \mathbb{Q}[t]/(t) \to H^i(L_\bullet) \otimes \mathbb{Q}[t]/(t) \to 0,$$

and in the fourth line that $\dim(N\otimes \mathbb{Q}[t]/(t)) = \dim \mathrm{Tor}_1^{\mathbb{Q}[t]}(N,\mathbb{Q}[t]/(t))$ for a torsion $\mathbb{Q}[t]$-module N. □

3.3. **Proof of theorem 3.1.** We proceed to the proof of the Hilali conjecture for elliptic spaces of pure type. We have to prove that

$$\dim H^*(\Lambda V,d)\geq 2n+r.$$

By proposition 3.3, we need to prove that $\dim \mathrm{Tor}^*_S(M,\mathbb{Q}_0)\geq 2n+r$. Consider the family

$$\mathcal{M}=\frac{\mathbb{Q}[t,x_1,\ldots,x_n]}{(P_1+tx_1,\ldots,P_n+tx_n)}\,.$$

For small t, the hypersurfaces $P_1+tx_1,\ldots,P_n+tx_n$ intersect in N points near the origin accounted with multiplicity, where $N=\mathrm{length}(M)$. Therefore $\mathcal{M}$ is a flat family. By lemma 3.5, it is enough to bound below $\dim \mathrm{Tor}^*_S(M_\xi,\mathbb{Q}_0)$. But for generic t, the hypersurfaces $P_1+tx_1,\ldots,P_n+tx_n$ intersect in N distinct points (at least, it is clear that they intersect in several points and the origin is isolated of multiplicity one). Therefore

$$\mathrm{Tor}^k_S(M_\xi,\mathbb{Q}_0)=\mathrm{Tor}^k_S(\mathbb{Q}_0,\mathbb{Q}_0)\,.$$

This is easily computed to have dimension $\binom{r}{k}$ (using the Koszul complex). So, using also lemma 3.4,

$$\begin{aligned}\dim \mathrm{Tor}^*_S(M,\mathbb{Q}_0) &\geq (n+1)+\sum_{k=1}^{r-1}\dim \mathrm{Tor}^k_S(M,\mathbb{Q}_0)+(n+1)\\ &\geq 2n+2+\sum_{k=1}^{r-1}\dim \mathrm{Tor}^k_S(M_\xi,\mathbb{Q}_0)\\ &= 2n+2+\sum_{k=1}^{r-1}\binom{r}{k}=2n+2^r\geq 2n+r\,.\end{aligned}$$

Remark 3.6. The above computation works for $r\geq 1$. If $r=0$ then we have to prove that $\mathrm{length}(M)\geq 2n$. But then computing the degree 2 non-zero elements in M, we have that they are at least $\binom{n+1}{2}-n$. So for any n,

$$\mathrm{length}(M)\geq 1+n+\binom{n+1}{2}-n=\frac{1}{2}(n+1)n+1\geq 2n.$$

4. The hyperelliptic case

A minimal model $(\Lambda V,d)$ of elliptic type is *hyperelliptic* if $V=V^{even}\oplus V^{odd}$, and

$$d(V^{even})=0,\quad d(V^{odd})\subset \Lambda^+V^{even}\otimes \Lambda V^{odd}\,. \tag{7}$$

An elliptic space is hyperelliptic if its minimal model is so. Note that elliptic spaces of pure type are in particular hyperelliptic.

By proposition 2.1 we have that $\dim V^{even}-\dim V^{odd}\leq 0$. Let $n=\dim V^{even}$ and $n+r=\dim V^{odd}$, where $r\geq 0$. Write $x_1,\ldots,x_n$ for the generators of even degree, and $y_1,\ldots,y_{n+r}$ for the generators of odd degree. Then $dx_i=0$, and $dy_j=P_j(x_1,\ldots,x_n,y_1,\ldots,y_{j-1})$, where P_j do not have linear terms.

In this section we prove the following:

Theorem 4.1. *The Hilali conjecture holds for hyperelliptic spaces.*

4.1. Semicontinuity for elliptic minimal models.

Lemma 4.2. *Let V be a graded rational finite-dimensional vector space, and let d be a differential for $\Lambda V \otimes \mathbb{Q}[t]$ such that $dt = 0$, where t has degree 0. Take a non-numerable field $\mathbf{k} \supset \mathbb{Q}$, $V_{\mathbf{k}} = V \otimes \mathbf{k}$. We denote by d_ξ the differential induced in $\Lambda V_{\mathbf{k}} = \Lambda V \otimes \mathbf{k}[t]/(t-\xi)$, for $\xi \in \mathbf{k}$. Then*

$$\dim H(\Lambda V_{\mathbf{k}}, d_\xi) \le \dim H(\Lambda V, d_0)\,,$$

for generic $\xi \in \mathbf{k}$.

Proof. Write

$$0 \to \tilde{K} \to \Lambda V \otimes \mathbf{k}[t] \to \tilde{I} \to 0\,,$$

where $\tilde{K}$ and $\tilde{I}$ are the kernel and image of d, resp. Note that both $\tilde{K}$ and $\tilde{I}$ are free $\mathbf{k}[t]$-modules, being submodules of $\Lambda V \otimes \mathbf{k}[t]$.

Denote by $\mathbf{k}_\xi = \mathbf{k}[t]/(t-\xi)$. Then we have a diagram

$$\begin{array}{ccccccccc} 0 & \to & \tilde{K} \otimes \mathbf{k}_\xi & \to & (\Lambda V \otimes \mathbf{k}[t]) \otimes \mathbf{k}_\xi & \to & \tilde{I} \otimes \mathbf{k}_\xi & \to & 0 \\ & & \downarrow & & || & & \downarrow & & \\ 0 & \to & K & \to & \Lambda V_{\mathbf{k}} & \to & I & \to & 0. \end{array} \tag{8}$$

(Here the tensor products of all $\mathbf{k}[t]$-modules are over $\mathbf{k}[t]$, and the tensor product $\Lambda V \otimes \mathbf{k}[t]$ is over the rationals.) Therefore the last vertical map is a surjection, and the first map is an injection.

We have

$$0 \to \tilde{I} \to \tilde{K} \to H(\Lambda V \otimes \mathbf{k}[t], d) \to 0\,,$$

which is an exact sequence of $\mathbf{k}[t]$-modules. Then $H(\Lambda V \otimes \mathbf{k}[t], d)$ contains a free part and a torsion part. The torsion is supported at some points, which are at most countably many. Therefore for generic $\xi \in \mathbf{k}$,

$$0 \to \tilde{I} \otimes \mathbf{k}_\xi \to \tilde{K} \otimes \mathbf{k}_\xi \to H(\Lambda V \otimes \mathbf{k}[t], d) \otimes \mathbf{k}_\xi \to 0$$

is exact. As $\tilde{I} \otimes \mathbf{k}_\xi \twoheadrightarrow I \subset K$ and $\tilde{I} \otimes \mathbf{k}_\xi \subset \tilde{K} \otimes \mathbf{k}_\xi \subset K$, we have that the last map in (8) is an injection, therefore an isomorphism, thus first map is also an isomorphism by the snake lemma.

Note that also, when tensoring with $\mathbf{k}(t)$, we have an exact sequence

$$0 \to \tilde{I} \otimes \mathbf{k}(t) \to \tilde{K} \otimes \mathbf{k}(t) \to H(\Lambda V \otimes \mathbf{k}[t], d) \otimes \mathbf{k}(t) \to 0\,.$$

Also $H(\Lambda V \otimes \mathbf{k}[t], d) \otimes \mathbf{k}(t) = H(\Lambda V \otimes \mathbf{k}(t), d)$, since $\mathbf{k}(t)$ is a flat $\mathbf{k}[t]$-module. Hence

$$\begin{aligned} \dim H(\Lambda V_{\mathbf{k}}, d_\xi) &= \dim K - \dim I \\ &= \dim \tilde{K} \otimes \mathbf{k}_\xi - \dim \tilde{I} \otimes \mathbf{k}_\xi \\ &= \dim H(\Lambda V \otimes \mathbf{k}(t), d)\,. \end{aligned}$$

In the first line, we mean $\dim K - \dim I = \sum_{d \ge 0} (\dim K^d - \dim I^d)$.

Take now $\xi = 0$. The map $\tilde{K} \to K \to K/I$ factors as $\tilde{K}/\tilde{I} \to K/I$. Tensor this map by $\mathbf{k}_0$ to get $(\tilde{K}/\tilde{I}) \otimes \mathbf{k}_0 \to K/I$. Note that there is an exact sequence

$$\tilde{I} \otimes \mathbf{k}_0 \to \tilde{K} \otimes \mathbf{k}_0 \to (\tilde{K}/\tilde{I}) \otimes \mathbf{k}_0 \to 0,$$

but the first map may not be injective. Then there is a map

$$\frac{\tilde{K}\otimes \mathbf{k}_0}{\mathrm{Im}(\tilde{I}\otimes \mathbf{k}_0)} = (\tilde{K}/\tilde{I})\otimes \mathbf{k}_0 \to K/I\,.$$

By (8), this is an inclusion. Now we have:

$$\begin{aligned}\dim H(\Lambda V, d_\xi) &= \dim H(\Lambda V\otimes \mathbf{k}(t), d)\\ &= \dim(\tilde{K}/\tilde{I})\otimes \mathbf{k}(t)\\ &\le \dim(\tilde{K}/\tilde{I})\otimes \mathbf{k}_0\\ &= \dim \frac{\tilde{K}\otimes \mathbf{k}_0}{\mathrm{Im}(\tilde{I}\otimes \mathbf{k}_0)}\\ &\le \dim K/I\\ &= \dim H(\Lambda V_{\mathbf{k}}, d_0)\\ &= \dim_{\mathbb{Q}} H(\Lambda V, d_0)\,.\end{aligned}$$

□

4.2. **Perturbing the minimal model.** Let $x_1,\dots,x_n$ denote generators for V^{even}, and $y_1,\dots,y_{n+r}$ generators for V^{odd}. Here $dx_i=0$ and $dy_j=P_j(x_1,\dots,x_n,y_1,\dots,y_{j-1})$.

We consider the algebra

$$(\Lambda W, d) = (\Lambda V, d)\otimes(\Lambda \bar{y}_1, 0)\,,$$

where $\deg(\bar{y}_1)=\deg(x_1)-1$. Then

$$\dim H(\Lambda W, d) = 2\dim H(\Lambda V, d)\,.$$

Consider now the differential δ on ΛW such that $\delta x_j=0$, $\delta y_j=0$ and $\delta\bar{y}_1=x_1$. Hence $\delta^2=0$ and $d\delta=\delta d=0$. So

$$d_t = d + t\delta$$

is a differential on $\Lambda W\otimes \mathbf{k}[t]$.

For generic $\xi\in\mathbf{k}$, $(\Lambda W_{\mathbf{k}}, d_\xi)$ verifies that $d_\xi\bar{y}_1=\xi x_1$. So for non-zero ξ, there is a KS-extension [8, §1.4]

$$(\Lambda(x_1,\bar{y}_1), d_\xi)\longrightarrow(\Lambda W_{\mathbf{k}}, d_\xi)\longrightarrow(\Lambda(x_2,\dots,x_n,y_1,\dots y_{n+r}), d)\,.$$

As $H(\Lambda(x_1,\bar{y}_1), d_\xi)=\mathbf{k}$, we have that

$$H(\Lambda W_{\mathbf{k}}, d_\xi)\cong H(\Lambda(x_2,\dots,x_n,y_1,\dots y_{n+r}), d)\,.$$

Now we apply lemma 4.2 to this to obtain that

$$\dim H(\Lambda(x_2,\dots,x_n,y_1,\dots y_{n+r}), d)\le \dim H(\Lambda W, d) = 2\dim H(\Lambda V, d)\,.$$

Repeating the argument n times, we get that

$$\dim H(\Lambda(y_1,\dots y_{n+r}), d)\le 2^n\dim H(\Lambda V, d)\,.$$

But the hyperelliptic condition says that $d=0$ for the first space, so

$$2^n\dim H(\Lambda V, d)\ge \dim H(\Lambda(y_1,\dots y_{n+r}), d) = 2^{n+r}\,.$$

This gives

$$\dim H(\Lambda V, d)\ge 2^r\,. \tag{9}$$

4.3. **Another proof of (9).** In this paragraph we present a different proof of the inequality $\dim H(\Lambda V, d) \geq 2^r$ for hyperelliptic spaces. Recall that if A is a commutative graded differential algebra, and if M, N are differential graded A-modules, the differential Tor is defined as:

$$\mathrm{Tor}^*(M,N) = H^*(P \otimes_A N),$$

where $P \xrightarrow{\sim} M$ is a semifree resolution, i.e., a quasi-isomorphism from a semifree A-module P to M (see [2, §6]).

Lemma 4.3. *Let $C \xleftarrow{\varphi} A \xrightarrow{\psi} B$ be morphisms of commutative differential graded algebras. There exists a convergent spectral sequence:*

$$E_2^{p,q} = H^p(B) \otimes \mathrm{Tor}_A^q(\mathbb{Q}, C) \Rightarrow \mathrm{Tor}_A^{p+q}(B, C).$$

Proof. Decompose φ and ψ as:

$$\begin{array}{ccc} A & \longrightarrow & A \otimes \Lambda W \\ & {\scriptstyle\psi}\searrow & \downarrow {\scriptstyle\alpha}\,\wr \\ & & B \end{array} \qquad\qquad \begin{array}{ccc} A & \longrightarrow & A \otimes \Lambda U \\ & {\scriptstyle\varphi}\searrow & \downarrow {\scriptstyle\beta}\,\wr \\ & & C \end{array}$$

Then $\alpha \colon A \otimes \Lambda W \xrightarrow{\sim} B$ is a semifree resolution of B regarded as A-module, so

$$\mathrm{Tor}_A^*(B, C) = H^*((A \otimes \Lambda W) \otimes_A C).$$

Moreover, $\mathrm{Id} \otimes \beta \colon (A \otimes \Lambda W) \otimes_A A \otimes \Lambda U \xrightarrow{\sim} (A \otimes \Lambda W) \otimes_A C$ is a quasi-isomorphism and $(A \otimes \Lambda W) \otimes_A (A \otimes \Lambda U) \cong A \otimes \Lambda W \otimes \Lambda U$. Therefore one gets a rational fibration

$$A \otimes \Lambda W \to A \otimes \Lambda W \otimes \Lambda U \to \Lambda U,$$

whose associated Serre spectral sequence has the form

$$E_2^{p,q} = H^p(A \otimes \Lambda W) \otimes H^q(\Lambda U) \Rightarrow H^{p+q}(A \otimes \Lambda W \otimes \Lambda U).$$

On the one hand, $H^*(A \otimes \Lambda W) = H^*(B)$. On the other hand, since β is a semifree resolution of C, we have that:

$$H^*(\Lambda U) = H^*((A \otimes \Lambda U) \otimes_A \mathbb{Q}) = \mathrm{Tor}_A^*(\mathbb{Q}, C).$$

Putting all pieces together we get

$$E_2^{p,q} = H^p(B) \otimes \mathrm{Tor}_A^q(\mathbb{Q}, C) \Rightarrow \mathrm{Tor}_A^{p+q}(B, C)$$

□

Theorem 4.4. *Let $(\Lambda V, d)$ be a hyperelliptic minimal model. Then*

$$\dim H(\Lambda V, d) \geq 2^r.$$

Proof. Write as usual $x_1, \ldots, x_n$ for generators of $X = V^{even}$ and $y_1, \ldots, y_{n+r}$ for generators of $Y = V^{odd}$. When we apply the previous lemma to morphisms $\mathbb{Q} \longleftarrow \Lambda X \hookrightarrow \Lambda V$ we get a spectral sequence:

$$E_2 = H(\Lambda V, d) \otimes \mathrm{Tor}_{\Lambda X}^*(\mathbb{Q}, \mathbb{Q}) \Rightarrow \mathrm{Tor}_{\Lambda X}^*(\Lambda V, \mathbb{Q}).$$

On the one hand,

$$\mathrm{Tor}_{\Lambda X}^*(\mathbb{Q}, \mathbb{Q}) = H^*(\Lambda(\overline{x}_1, ..., \overline{x}_n), 0) = \Lambda(\overline{x}_1, \ldots, \overline{x}_n),$$

where $\Lambda(x_1,...,x_n,\overline{x}_1,...,\overline{x}_n) \xrightarrow{\sim} \mathbb{Q}$ is a semifree resolution of $\mathbb{Q}$ regarded as ΛX-module. Hence $\overline{x}_i$ are all of odd degree.

On the other hand, ΛV is already ΛX-semifree, so:

$$\mathrm{Tor}^*_{\Lambda X}(\Lambda V,\mathbb{Q}) = H(\Lambda V \otimes_{\Lambda X} \mathbb{Q}) = H^*(\Lambda(y_1,...,y_{n+k}),0) = \Lambda(y_1,...,y_{n+k}).$$

Then the inequality

$$\dim H^*(\Lambda V,d)\cdot \dim \mathrm{Tor}^*_{\Lambda X}(\mathbb{Q},\mathbb{Q}) \geq \dim \mathrm{Tor}^*_{\Lambda X}(\Lambda V,\mathbb{Q})$$

coming from the spectral sequence translates into

$$2^n \dim H^*(\Lambda V,d) \geq 2^{n+r},$$

so the result follows. □

4.4. **Proof of theorem 4.1.** Now we prove the inequality $\dim H(\Lambda V,d) \geq 2n+r$, for the hyperelliptic minimal model.

If $r=0$, then $\chi_\pi = 0$. So [2, Prop. 32.10] says that the model is pure, and this case is already covered by remark 3.6.

If $r>0$, then $\chi_\pi<0$. So by proposition 2.1, $\chi=0$, and hence it is enough to prove that

$$\dim H^{even}(\Lambda V,d) \geq n+\tfrac{r}{2}.$$

Suppose that $r=1,2$. As the degree 0 and degree 1 elements give always non-trivial homology classes, then $\dim H^{even}(\Lambda V,d)\geq n+1$, and we are done.

So we can assume $r\geq 3$. We use the following fact: if $P(x)$ is a quadratic polynomial on the x, and $P(x)=d\alpha$, $\alpha\in\Lambda V$, then α must be linear, $\alpha\in V^{odd}$ and denoting by d_o the composition

$$V^{odd} \longrightarrow \Lambda^+V^{even}\otimes\Lambda V^{odd} \twoheadrightarrow \Lambda^+V^{even},$$

we have $P(x)=d_o\alpha$. So there are at least $\binom{n+1}{2}-(n+r)$ quadratic terms in the homology. Conjecture 1.1 is proved if

$$\left\{\begin{array}{rl} \text{either} & 1+n+\binom{n+1}{2}-(n+r)\geq n+\frac{r}{2}\,, \\ \text{or} & 2^r \geq 2n+r\,. \end{array}\right. \tag{10}$$

So now assume that (10) does not hold. Then

$$2^r - r \leq 2n-1\,, \tag{11}$$

and $1+\binom{n+1}{2}-n<\frac{3}{2}r$, i.e.,

$$(2n-1)^2 \leq 12r-11. \tag{12}$$

Putting together (11) and (12), we get $2^r-r\leq\sqrt{12r-11}$, i.e., $2^r\leq r+\sqrt{12r-11}$. This is easily seen to imply that $r\leq 3$. So $r=3$ and $n=3$.

There remains to deal with the case $n=3$, $r=3$, and d_o is an isomorphism of the odd degree elements onto Λ^2V^{even}. Let x_1,x_2,x_3 be the even degree generators, of degrees $d_1\leq d_2\leq d_3$ respectively. The degrees of $x_1^2, x_1x_2, x_2^2, x_1x_3, x_2x_3, x_3^2$ are the six numbers

$$2d_1 \leq d_1+d_2 \leq 2d_2, \quad d_1+d_3\leq d_2+d_3\leq 2d_3.$$

We have two cases:

- Case $2d_2 \leq d_1 + d_3$. We can arrange the odd generators $y_1, \dots, y_6$ with increasing degree and so that $d_o y_1 = x_1^2, d_o y_2 = x_1 x_2, d_o y_3 = x_2^2, d_o y_4 = x_1 x_3, d_o y_5 = x_2 x_3, d_o y_6 = x_3^2$. Clearly, $dy_1 = x_1^2$. Then $dy_2 = x_1 x_2 + P(x_1)$, where $P(x_1)$ is a polynomial on x_1, i.e., of the form cx_1^n, $n \geq 2$. But this can absorbed by a change of variables $y_2 \mapsto y_2 - cx_1^{n-2} y_1$. So we can write $dy_2 = x_1 x_2$. Now the even-degree closed elements in $\Lambda(x_1, x_2, x_3, y_1, y_2)$ are again polynomials on x_1, x_2, x_3. So we can assume $dy_3 = x_2^2$ as before. Continuing the computation, the even-degree closed elements in $\Lambda(x_1, x_2, x_3, y_1, y_2, y_3)$ are either polynomials on the x_i's or a multiple of the element $x_2^2 y_1 y_2 - x_1 x_2 y_1 y_3 + x_1^2 y_2 y_3 = d(y_1 y_2 y_3)$, which is exact. So we can again manage to arrange that $dy_4 = x_1 x_3$.
- Case $2d_2 > d_1 + d_3$. Then we have that $d_o y_3 = x_1 x_3$ and $d_o y_4 = x_2^2$. As before, we can arrange $dy_3 = x_1 x_3$. Now the even-degree closed elements in $\Lambda(x_1, x_2, x_3, y_1, y_2, y_3)$ are polynomials on the x_i's or a multiple of $x_3 y_1 y_2 - x_2 y_1 y_3 + x_1 y_2 y_3$. But this element has degree $3d_1 + d_2 + d_3 - 2 > 2d_2$, so it must be $dy_4 = x_2^2$.

In either case, dy_1, dy_2, dy_3, dy_4 are $x_1^2, x_1 x_2, x_2^2$ and $x_1 x_3$. Let us assume that we are in the first case to carry over the notation.

Now we compute the even-degree closed elements in $\Lambda(x_1, x_2, x_3, y_1, y_2, y_3, y_4)$. These are polinomials on x_i's or combinations of

$$
\begin{aligned}
& x_2^2 y_1 y_2 - x_1 x_2 y_1 y_3 + x_1^2 y_2 y_3 = d(y_1 y_2 y_3), \\
& x_3 y_1 y_2 - x_2 y_1 y_4 + x_1 y_2 y_4, \\
& x_1 x_3 y_2 y_3 - x_2^2 y_2 y_4 + x_1 x_2 y_3 y_4 = d(y_2 y_3 y_4), \text{ and} \\
& x_1 x_3 y_1 y_3 + x_1^2 y_3 y_4 - x_2^2 y_1 y_4 = d(y_1 y_3 y_4).
\end{aligned}
$$

Only the second one is non-exact, but its degree is strictly bigger thatn $d_2 + d_3$. So again we can arrange that $dy_5 = x_2 x_3$.

Finally, the minimal model is:

$$
\begin{cases}
dy_1 = x_1^2, \\
dy_2 = x_1 x_2, \\
dy_3 = x_2^2, \\
dy_4 = x_1 x_3, \\
dy_5 = x_2 x_3, \\
dy_6 = x_3^2 + P(x_i, y_j).
\end{cases}
$$

The even-degree closed elements in $\Lambda(x_1, x_2, x_3, y_1, y_2, y_3, y_4, y_5)$ contain at least

$$
\begin{aligned}
\alpha_1 &= x_3 y_2 y_3 + x_1 y_3 y_5 - x_2 y_2 y_5 \,, \\
\alpha_2 &= x_3 y_1 y_2 - x_2 y_1 y_4 + x_1 y_2 y_4 \,.
\end{aligned}
$$

At most one of them does not survive in $H(\Lambda V, d)$, so proving the existence of at least another even-degree cohomology class. Hence $\dim H(\Lambda V, d) \geq 10 \geq 9$, as required.

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